Personal Solutions to

Category Theory in Context

by Emily Riehl

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Available online at
https://gitlab.com/cionx/solutions-category-theory-in-context-riehl.

Preface

The following are my solutions to the exercises in Emily Riehl's textbook *Category Theory in Context*.

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Chapter 1

Categories, Functors, Natural Transformations

1.1 Abstract and concrete categories

Exercise 1.1.i

(i)

We find that

$$g = g1_x = gfh = 1_y h = h\,.$$

It follows from this equality $g = h$ that the morphism g is not only a left-sided inverse to the morphism f, but also a right-sided inverse, because h is a right-sided inverse to f. Therefore, g is a two-sided inverse to f. The existence of this two-sided inverse means precisely that f is an isomorphism.

(ii)

Suppose that a morphism $f : x \to y$ in a category C admits two inverses g and h. This means, more explicitly, that both g and h are two-sided inverse to f. This entails that $gf = 1_x$ and also $fh = 1_y$. According to the previous part of this exercise, we thus have $g = h$.

Exercise 1.1.ii

We denote the suspected category by G. It has the same objects as the original category C, but its morphisms are only the isomorphisms from C.

The objects of G are also objects of C by definition of G, and for every two objects x and y of G, the set $G(x, y)$ of isomorphisms from x to y (in C) is a subset of $C(x, y)$.

For every object x of C let 1_x be the identity morphism of x in the original category C. This identity morphism is an isomorphism in C (it is its own inverse) and therefore contained in G. This shows that all identity morphisms from C are contained in G.

Let f and g be two morphisms in G that are composable in C, i.e., such that the codomain of f equals the domain of g; suppose more specifically that $f : x \to y$ and $g : y \to z$. Both f and g are isomorphisms in C, and the composite gf in C is therefore again an isomorphism in C: its inverse is given by the composite $f^{-1}g^{-1}$. Therefore, gf is again contained in G.

We have shown that G contains all identity morphisms of C and that G is closed under composition of morphisms. This shows that G is a subcategory of C.

Let $f : x \to y$ be a morphism in G. This means that f is in isomorphism in C, which in turn means that there exists a (unique) morphism $f^{-1} : y \to x$ with both $ff^{-1} = 1_y$ and $f^{-1}f = 1_x$ in C. But these equalities also tell us that f^{-1} is an isomorphism with inverse f (so that $(f^{-1})^{-1} = f$), which entails that f^{-1} is also a morphism in G. The morphisms f and f^{-1} are also mutually inverse in G, because G is a subcategory of C. Therefore, f is an isomorphism in G.

We have shown that every morphism in G is an isomorphism, not only in C but already in G, which shows that G is a groupoid.

Let now G′ be another subcategory of C that is also a groupoid. This means that there exists for every morphism $f : x \to y$ in G′ another morphism $f^{-1} : y \to x$ in G′ with $ff^{-1} = 1_y$ and $f^{-1}f = 1_x$ in G′. In these two equalities, both composition and identity morphisms take place in G′. But since G′ is a subcategory of G, these equations entail that f and f^{-1} are also mutually inverse in C. In other words, f is an isomorphism in C. This in turn means that f is contained in G.

We have shown that every morphism in G′ is also contained in G. This shows that G is indeed the *maximal* groupoid in C.

Exercise 1.1.iii

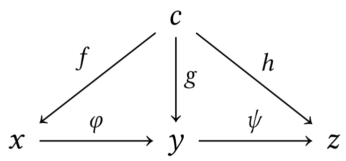
(i)

We denote the object of c/C as pairs (f, x), where x is an object of C and f is a morphism from c to x in C.

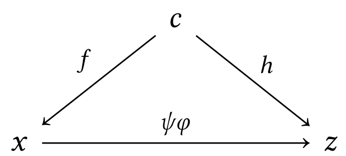
We first need to explain how the composition of morphisms in c/C is supposed to work. Given two such morphisms

$$\varphi\colon (f, x) \longrightarrow (g, y) \quad \text{and} \quad \psi\colon (g, y) \longrightarrow (h, z)$$

in c/C, we have the following commutative diagram:



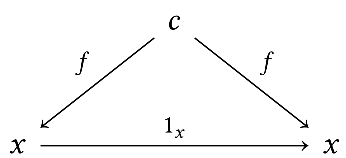
By leaving the node y out of this diagram, we arrive at the following commutative diagram:



The commutativity of this diagram tells us that the composite $\psi\varphi$ (taken in C) is a morphism from (f, x) to (h, z) in c/C. This observation allows us to define the composite of φ and ψ in c/C as their composite in C.

The associativity of composition of morphisms in c/C follows from the associativity of composition of morphisms in C.

We have for every object (f, x) in c/C the following commutative diagram:



The commutativity of this diagram tells us that 1_x is a morphism from (f, x) to (f, x) in c/C. We have for every morphism $\varphi\colon (f, x) \to (g, y)$ in c/C the equalities

$$\varphi 1_x = \varphi\,, \quad \text{and} \quad 1_y \varphi = \varphi$$

in C, and therefore also in c/C.

We have thus shown that the identity morphism of x in C serves as the identity morphism of (f, x) in c/C.[1]

(ii)

This part of the exercise works completely dual to the first part: just reverse all the arrows. (That is, $\mathsf{C}/c \cong (c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$.)

1.2 Duality

Exercise 1.2.i

Let c be an object in a category C.

The exercise tasks us with proving that two categories are isomorphic, even though the book has yet to introduce the notion of an isomorphism of categories. (We haven't even introduced functors yet.) We will construct in the following a contravariant functor from C/c to $c/\mathsf{C}^{\mathrm{op}}$ that is bijective on objects and on morphisms.

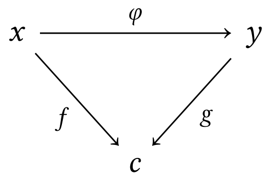
The objects of the slice category C/c are the pairs (x, f) consisting of another object x of C and a morphism $f : x \to c$ in C. Similarly, the objects of the slice category $c/\mathsf{C}^{\mathrm{op}}$ are the pairs (x, f') consisting of another object x of C^{op} and a morphism $f' : c \to x$ in C^{op}. The two categories C and C^{op} have the same objects, and we know that morphisms in C^{op} correspond bijectively to morphisms in C via the mapping $(f : x \to y) \mapsto (f^{\mathrm{op}} : y \to x)$. It follows that this bijection restricts to a bijection between the objects of C/c on the one hand and the objects of $c/\mathsf{C}^{\mathrm{op}}$ on the other hand. More explicitly, this bijection is given by

$$(x, f : x \to c) \longmapsto (x, f^{\mathrm{op}} : c \to x) .$$

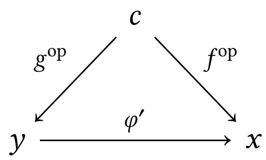
Let (x, f) and (y, g) be two objects of C/c. A morphism from (x, f) to (y, g) in C/c is a morphism $\varphi : x \to y$ in the category C for which the following

[1] We should actually ensure that Hom-spaces in c/C are pairwise disjoint, but we don't care about this for now – we could just apply the usual construction of replacing every morphism $\varphi : (f, x) \to (g, y)$ by its triple $(\varphi, (f, x), (g, y))$ to ensure this technical requirement.

diagram commutes:



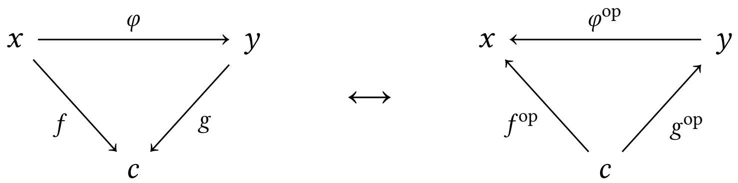
Similarly, a morphism from (y, g^{op}) to (x, f^{op}) in the category $c/\mathsf{C}^{\mathrm{op}}$ is a morphism $\varphi' : y \to x$ in C^{op} for which the following diagram commutes:



We have therefore for every morphism $\varphi : x \to y$ in C the sequence of equivalences

$$
\begin{aligned}
& \varphi \text{ is a morphism from } (x, f) \text{ to } (y, g) \text{ in } \mathsf{C}/c \\
\iff\ & g\varphi = f \\
\iff\ & (g\varphi)^{\mathrm{op}} = f^{\mathrm{op}} \\
\iff\ & \varphi^{\mathrm{op}} g^{\mathrm{op}} = f^{\mathrm{op}} \\
\iff\ & \varphi^{\mathrm{op}} \text{ is a morphism from } (y, g^{\mathrm{op}}) \text{ to } (x, f^{\mathrm{op}}) \text{ in } c/\mathsf{C}^{\mathrm{op}} .
\end{aligned}
$$

These equivalences tell us that the bijection between morphisms of C and morphisms of C^{op} given by $\varphi \mapsto \varphi^{\mathrm{op}}$ restricts for every two objects (x, f) and (y, g) of C/c to a bijection between morphisms from (x, f) to (y, g) in C/c on the one hand and morphisms from (y, g^{op}) to (x, f^{op}) in $c/\mathsf{C}^{\mathrm{op}}$ on the other hand:



We have thus constructed a bijection between the objects of the two categories C/c and $c/\mathsf{C}^{\mathrm{op}}$, given by $(x, f) \mapsto (x, f^{\mathrm{op}})$, as well as for every two objects (x, f) and (y, g) of C/c a bijection between the morphisms from (x, f)

to (y, g) in C/c and the morphisms from (y, g^{op}) to (x, f^{op}) in $c/\mathsf{C}^{\mathrm{op}}$, given by $\varphi \mapsto \varphi^{\mathrm{op}}$. We denote these mappings by D (for "duality").

It remains to check the functoriality of D. We have for every object (x, f) of C/c the sequence of equalities

$$D(1_{(x,f)}) = 1_{(x,f)}^{\mathrm{op}} = 1_{x,\mathsf{C}}^{\mathrm{op}} = 1_{x,\mathsf{C}^{\mathrm{op}}} = 1_{(x,f^{\mathrm{op}})} = 1_{D((x,f))},$$

which shows that D preserves identities. We also have for any two composable morphisms $\varphi : (x, f) \to (y, g)$ and $\psi : (y, g) \to (z, h)$ in C/c the sequence of equalities

$$D(\psi\varphi) = (\psi\varphi)^{\mathrm{op}} = \varphi^{\mathrm{op}}\psi^{\mathrm{op}} = D(\varphi)D(\psi),$$

which shows that D contravariantly preserves composition.

Regarding the second part of this exercise: by using the contravariant isomorphism D, one could actually define C/c as $(c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$. This would then allow us to deduce part (ii) of Exercise 1.1.iii from the previous part (i).

Exercise 1.2.ii

(i)

Suppose first that f is a split epimorphism. This means that there exists a morphism $g : y \to x$ with $fg = 1_y$. It follows for every object c of C for the two induced functions

$$f_* : \mathsf{C}(c, x) \longrightarrow \mathsf{C}(c, y) \quad \text{and} \quad g_* : \mathsf{C}(c, y) \longrightarrow \mathsf{C}(c, x)$$

that

$$f_*(g_*(\varphi)) = f_*(g\varphi) = fg\varphi = 1_y\varphi = \varphi$$

for every element φ of the set $\mathsf{C}(c, y)$. This shows that the function g_* is right inverse to the function f_*. The existence of such a right-inverse entails that f_* is surjective.

Suppose now that the induced function $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is surjective for every object c of C. By choosing c as y, we can see that the map

$$f_* : \mathsf{C}(y, x) \longrightarrow \mathsf{C}(y, y), \quad g \longmapsto fg$$

is surjective. This surjectivity entails that there exists an element g of $\mathsf{C}(y, x)$ – that is, a morphism $g : y \to x$ in C – such that $fg = 1_y$. The existence of this morphism g tells us that f is split epimorphism.

(ii)

We have the following sequence of equivalences:

$$
\begin{aligned}
& f : x \to y \text{ is a split monomorphism in } \mathsf{C} \\
\iff & \text{there exists a morphism } g : y \to x \text{ in } \mathsf{C} \text{ with } gf = 1_{x,\mathsf{C}} \\
\iff & \text{there exists a morphism } g : y \to x \text{ in } \mathsf{C} \text{ with } (gf)^{\mathrm{op}} = 1_{x,\mathsf{C}}^{\mathrm{op}} \\
\iff & \text{there exists a morphism } g : y \to x \text{ in } \mathsf{C} \text{ with } f^{\mathrm{op}} g^{\mathrm{op}} = 1_{x,\mathsf{C}^{\mathrm{op}}} \\
\iff & \text{there exists a morphism } g' : x \to y \text{ in } \mathsf{C} \text{ with } f^{\mathrm{op}} g' = 1_{x,\mathsf{C}^{\mathrm{op}}} \\
\iff & f^{\mathrm{op}} : y \to x \text{ is a split epimorphism in } \mathsf{C}^{\mathrm{op}} \\
\iff & (f^{\mathrm{op}})_* : \mathsf{C}^{\mathrm{op}}(c, y) \to \mathsf{C}^{\mathrm{op}}(c, x) \text{ is surjective for every object } c \text{ of } \mathsf{C}^{\mathrm{op}} \\
\iff & (f^{\mathrm{op}})_* : \mathsf{C}^{\mathrm{op}}(c, y) \to \mathsf{C}^{\mathrm{op}}(c, x) \text{ is surjective for every object } c \text{ of } \mathsf{C}\,.
\end{aligned}
$$

We observe that the diagram

$$
\begin{array}{ccc}
\mathsf{C}(y, c) & \xrightarrow{\;f^*\;} & \mathsf{C}(x, c) \\
{\scriptstyle (-)^{\mathrm{op}}} \big\downarrow & & \big\downarrow {\scriptstyle (-)^{\mathrm{op}}} \\
\mathsf{C}^{\mathrm{op}}(c, y) & \xrightarrow{\;(f^{\mathrm{op}})_*\;} & \mathsf{C}^{\mathrm{op}}(c, x)
\end{array}
$$

commutes because

$$f^*(h)^{\mathrm{op}} = (hf)^{\mathrm{op}} = f^{\mathrm{op}} h^{\mathrm{op}} = (f^{\mathrm{op}})_*(h^{\mathrm{op}})$$

for every element h of $\mathsf{C}(y, c)$. Both vertical arrows in this diagram are bijections. Consequently, the upper horizontal arrow is surjective if and only if the lower horizontal arrow is surjective.

Exercise 1.2.iii

We prove (i) and (ii) by providing two proofs for each statement.

Statement (i), first proof

Let $h_1, h_2 : c \to x$ be two morphisms with $gfh_1 = gfh_2$. Then $fh_1 = fh_2$ because g is a monomorphism, and then furthermore $h_1 = h_2$ because f is a monomorphism. This shows that the composite gf is again a monomorphism.

Statement (i), second proof

Let c be an arbitrary object of C. By assumption, both maps

$$f_* : \mathrm{C}(c, x) \longrightarrow \mathrm{C}(c, y), \quad g_* : \mathrm{C}(c, y) \longrightarrow \mathrm{C}(c, z)$$

are injective. It follows that their composite $g_* f_*$ is again injective. But we have the identity $g_* f_* = (gf)_*$. We have thus found that the map

$$(gf)_* : \mathrm{C}(c, x) \longrightarrow \mathrm{C}(c, z)$$

is injective for every object c of C. This tells us that the composite gf is again a monomorphism.

Statement (ii), first proof

Let $h_1, h_2 : c \to x$ be two morphisms with $fh_1 = fh_2$. Then also $gfh_1 = gfh_2$, and thus $h_1 = h_2$ because gf is a monomorphism. This shows that f is a monomorphism.

Statement (ii), second proof

Let c be an arbitrary object of C. By assumption, the induced map

$$(gf)_* : \mathrm{C}(c, x) \longrightarrow \mathrm{C}(c, z)$$

is injective. But this induced function equals the composite $g_* f_*$, whence this composite is injective. It follows (from naive set theory) that the function f_* is injective. We have thus shown that the map

$$f_* : \mathrm{C}(c, x) \longrightarrow \mathrm{C}(c, y)$$

is injective for every object c of C. This tells us that the morphism f is a monomorphism.

Concluding (i')

We first observe for every morphism $f\colon x \to y$ in C the following sequence of equivalences:

$$
\begin{aligned}
& f\colon x \to y \text{ is an epimorphism in C} \\
\iff & \text{for all } g_1, g_2\colon y \to c \text{ in C, } g_1 f = g_2 f \text{ implies } g_1 = g_2 \\
\iff & \text{for all } g_1, g_2\colon y \to c \text{ in C, } (g_1 f)^{\mathrm{op}} = (g_2 f)^{\mathrm{op}} \text{ implies } g_1^{\mathrm{op}} = g_2^{\mathrm{op}} \\
\iff & \text{for all } g_1, g_2\colon y \to c \text{ in C, } f^{\mathrm{op}} g_1^{\mathrm{op}} = f^{\mathrm{op}} g_2^{\mathrm{op}} \text{ implies } g_1^{\mathrm{op}} = g_2^{\mathrm{op}} \\
\iff & \text{for all } g_1', g_2'\colon c \to y \text{ in } \mathrm{C}^{\mathrm{op}}, f^{\mathrm{op}} g_1' = f^{\mathrm{op}} g_2' \text{ implies } g_1' = g_2' \\
\iff & f^{\mathrm{op}}\colon y \to x \text{ is a monomorphism in } \mathrm{C}^{\mathrm{op}}\,.
\end{aligned}
$$

Thanks to this observation and part (i) of the lemma (i.e, Lemma 1.2.11), we can now observe the following sequence of equivalences:

$$
\begin{aligned}
& f\colon x \to y \text{ and } g\colon y \to z \text{ are epimorphism in C} \\
\iff & f^{\mathrm{op}}\colon y \to x \text{ and } g^{\mathrm{op}}\colon z \to y \text{ are monomorphisms in } \mathrm{C}^{\mathrm{op}} \\
\implies & f^{\mathrm{op}} g^{\mathrm{op}}\colon z \to x \text{ is a monomorphism in } \mathrm{C}^{\mathrm{op}} \\
\iff & (gf)^{\mathrm{op}}\colon z \to x \text{ is a monomorphism in } \mathrm{C}^{\mathrm{op}} \\
\iff & gf\colon x \to z \text{ is an epimorphism in } \mathrm{C}^{\mathrm{op}}
\end{aligned}
$$

This proves statement (i').

Concluding (ii')

By using part (ii) of the lemma and once again the above observation, we get the following sequence of equivalences:

$$
\begin{aligned}
& gf\colon x \to z \text{ is an epimorphism in C} \\
\iff & (gf)^{\mathrm{op}}\colon z \to x \text{ is a monomorphism in } \mathrm{C}^{\mathrm{op}} \\
\iff & f^{\mathrm{op}} g^{\mathrm{op}}\colon z \to x \text{ is a monomorphism in } \mathrm{C}^{\mathrm{op}} \\
\iff & g^{\mathrm{op}}\colon z \to y \text{ is a monomorphism in } \mathrm{C}^{\mathrm{op}} \\
\iff & g\colon y \to z \text{ is an epimorphism in C}\,.
\end{aligned}
$$

This proves statement (ii').

Monomorphisms form a subcategory

A class of morphism M in C forms a subcategory of C if and only if the following two conditions are satisfied:

1. M is closed under composition.
2. For every morphism $f : x \to y$ belonging to M, both 1_x and 1_y again belong to M.

In the case that M is supposed to define a full subcategory of C (i.e., a subcategory that contains all objects of C), the second condition can be simplified as follows:

2'. 1_x belongs to M for every object x of C.

We have already seen in Lemma 1.2.11 (and proven in the previous parts of this exercise) that the classes of monomorphisms and epimorphisms are both closed under composition. Identity morphisms are isomorphisms, therefore both split monomorphisms and also split epimorphisms, and therefore both monomorphisms and epimorphisms. Consequently, both the class of monomorphisms and the class of epimorphisms define full subcategories of C.

Exercise 1.2.iv

Every homomorphism of fields is injective, and therefore a monomorphism in Field.

Exercise 1.2.v

Let i denote the inclusion map from $\mathbb{Z}$ to $\mathbb{Q}$, which is a homomorphism of rings. The map i is injective, and therefore a monomorphism.

Let R be another ring and let f be a homomorphism of rings from $\mathbb{Q}$ to R. We have for every nonzero integer n the equalities

$$f(n) \cdot f\left(\frac{1}{n}\right) = f\left(n \cdot \frac{1}{n}\right) = f(1) = 1\,.$$

This shows that $f(1/n)$ is multiplicatively inverse to $f(n)$ in R. In other words, $f(n)$ is invertible in R and $f(1/n) = f(n)^{-1}$. It follows for every fraction p/q in $\mathbb{Q}$ that

$$f\left(\frac{p}{q}\right) = f\left(p \cdot \frac{1}{q}\right) = f(p) \cdot f\left(\frac{1}{q}\right) = f(p) \cdot f(q)^{-1} = (f \circ i)(p) \cdot (f \circ i)(q)^{-1}\,.$$

This shows that the homomorphism f is uniquely determined by its composite $f \circ i$. As this holds for every morphism f with domain $\mathbb{Q}$, we have shown that i is an epimorphism.

We have thus shown that i is both a monomorphism and an epimorphism. But it is not an isomorphism, as it is not bijective ($1/2$ does not lie in the image of i, whence i is not surjective).

We can more generally consider a commutative ring R, a multiplicative subset S of R, and the canonical homomorphism j from R to its localization $S^{-1}R$, given by $r \mapsto r/1$. Then the following hold:

- j is always an epimorphism.
- j is a monomorphism if and only if it is injective, which is the case if and only if S does not contain any zero divisor.
- j is an isomorphism if and only if every element of S is already invertible in R to begin with.

Exercise 1.2.vi

First part, first proof

Let $f : x \to y$ be a morphism in a category C that is both a monomorphism and a split epimorphism. The second assumption tells us that there exists a morphism $g : y \to x$ with $fg = 1_y$. It follows that $fgf = 1_y f = f = f 1_x$, and therefore $gf = 1_x$ because f is a monomorphism. This shows that g is already a two-sided inverse to f. The existence of this two-sided inverse means that f is an isomorphism.

First part, second proof

As before, let $f : x \to y$ be a morphism in a category C that is both a monomorphism and a split epimorphism. The induced map $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is injective for every object c of C because f is a monomorphism, and it is also surjective because f is a split epimorphism. This shows that f_* is a bijective for every object c of C, which in turn shows that f is an isomorphism.

Second part

Suppose now that f is a morphism in a category C that is both a split monomorphism and an epimorphism. This means that f^{op} is both a split epimorphism and a monomorphism in C^{op}. As seen above, f^{op} is therefore an isomorphism in C^{op}. This is in turn equivalent to f being an isomorphism in C.

Exercise 1.2.vii

Let A be a subset of a poset $(\mathsf{P}, \leq)$.

Definition of the supremum

An element s of P is a supremum of A if and only if for every element x of P the following condition holds: $x \geq s$ if and only if $x \geq a$ for every $a \in A$. In categorical terms, this condition means that for every object x of P, there exists a morphism $s \to x$ if and only if there exists a morphism $a \to x$ for every object a contained in A.

Definition of the infimum

Dually, an object i of P is an infimum of A if and only if for every object x of P, there exists a morphism $x \to i$ if and only if there exists a morphism $x \to a$ for every object a of A. That is, i is an infimum of A in P if and only if it is a supremum of A in P^{op}.

Uniqueness of the supremum

Let now s_1 and s_2 be two suprema of A. There exists a morphism $s_1 \to s_1$, namely the identity morphism. As s_1 is a supremum of A, there hence exists for every object a contained in A a morphism $a \to s_1$. As s_2 is a supremum of A, it follows that there exists a morphism $f : s_1 \to s_2$. By switching the roles of s_1 and s_2 we also find that there exists a morphism $g : s_2 \to s_1$.

The composite fg is a morphism $s_2 \to s_2$, and has therefore the same domain and codomain as the identity morphism 1_{s_2}. For any two objects x and y in P there exists at most one morphism from x to y, whence it follows that $fg = 1_{s_2}$. We find in the same way that also $gf = 1_{s_1}$.

This shows that f is an isomorphism with inverse g. But any two isomorphic objects of P are already equal (because the relation $\leq$ on P is antisymmetric). We hence find that $s_1 = s_2$.

Uniqueness of the infimum

Suppose that i_1 and i_2 are two infima of A. This means that i_1 and i_2 are suprema of A in P^{op}, whence $i_1 = i_2$.

1.3 Functoriality

Exercise 1.3.i

Let G and H be two groups. A functor Φ from $\mathsf{B}G$ to $\mathsf{B}H$ consists of a set-theoretic function $\{*\} \to \{*\}$ together with a set-theoretic function

$$\varphi : G \longrightarrow H$$

such that

$$\varphi(1_*) = 1_* \quad \text{and} \quad \varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$$

for all $g_1, g_2 \in G$. The second condition means precisely that φ is a homomorphism of groups, and this then entails the first condition.

We see that a functor from $\mathsf{B}G$ to $\mathsf{B}H$ is "the same" as a homomorphism of groups from G to H.

Exercise 1.3.ii

Let P and Q be two partially ordered sets, and let P and Q be the corresponding categories. A functor F from P to Q consists of a function

$$f : P \longrightarrow Q$$

as well as for every two elements p_1 and p_2 of P a function

$$F_{p_1,p_2} : \mathsf{P}(p_1, p_2) \longrightarrow \mathsf{Q}(f(p_1), f(p_2))$$

such that

$$F_{p,p}(1_p) = 1_{f(p)} \quad \text{and} \quad F_{p_1,p_3}(\psi\varphi) = F_{p_2,p_3}(\psi) \cdot F_{p_1,p_2}(\varphi) \tag{1.1}$$

for every object p in P and all morphisms $\varphi : p_1 \to p_2$ and $\psi : p_2 \to p_3$ in P.

We make two observations regarding the above data and conditions.

1. The sets $\mathsf{Q}(q_1, q_2)$ for $q_1, q_2 \in Q$ are either empty or singletons. The equalities (1.1) are therefore automatically satisfied.
2. The existence of a function $F_{p_1,p_2} \colon \mathsf{P}(p_1, p_2) \to \mathsf{Q}(f(p_1), f(p_2))$ is equivalent to the implication

$$\mathsf{P}(p_1, p_2) \neq \varnothing \implies \mathsf{Q}(f(p_1), f(p_2)) \neq \varnothing \,,$$

which is furthermore equivalent to the implication

$$p_1 \leq p_2 \implies f(p_1) \leq f(p_2) \,.$$

The existence of the functions F_{p_1,p_2} for all $p_1, p_2 \in P$ is therefore equivalent to f being isotone (i.e., weakly increasing).

We find altogether that a functor F from P to Q is “the same” as an isotone function f from P to Q.

Exercise 1.3.iii

We consider the partially ordered set P given by four elements a_1, b_1, a_2, b_2 with non-trivial relations $a_1 \leq b_1$ and $a_2 \leq b_2$, and the partially ordered set Q with elements $1, 2, 3$ and $1 \leq 2 \leq 3$. The categories P and Q corresponding to P and Q look respectively as follows:

$$P : \quad a_1 \longrightarrow b_1 \qquad a_2 \longrightarrow b_2$$

$$Q : \quad \begin{array}{ccccc} & & 2 & & \\ & \nearrow & & \searrow & \\ 1 & & \longrightarrow & & 3 \end{array}$$

We have a functor $F \colon \mathsf{P} \to \mathsf{Q}$ that maps the morphism $a_1 \to b_1$ to the morphism $1 \to 2$, and the morphism $a_2 \to b_2$ to the morphism $2 \to 3$. The two morphisms $1 \to 2$ and $2 \to 3$ lie in the image of F, but their composite $1 \to 3$ does not. The image of F is therefore not a subcategory of Q.

Exercise 1.3.iv

Let C be a category and let c be an object of C.

The covariant functor $\mathsf{C}(c,-)$

We need to show that $\mathsf{C}(c,-)$ is compatible with identities and composition of morphisms.

- Let x be an arbitrary object of C. The induced map
$$(1_x)_* : \mathsf{C}(c,x) \longrightarrow \mathsf{C}(c,x)$$
is the identity map of the set $\mathsf{C}(c,x)$ since
$$(1_x)_*(h) = 1_x \cdot h = h = 1_{\mathsf{C}(c,x)}(h)$$
for every $h \in \mathsf{C}(c,x)$.
- Let $f : x \to y$ and $g : y \to z$ be two composable morphisms in C. We have the sequence of equalities
$$(gf)_*(h) = (gf)h = g(fh) = g_*(fh) = g_*(f_*(h))$$
for every $h \in \mathsf{C}(c,x)$, and therefore the equality $(gf)_* = g_* f_*$.

We have altogether shown that $\mathsf{C}(c,-)$ is a covariant functor from C to Set.

The contravariant functor $\mathsf{C}(-,c)$

We proceed dually to the contravariant case.

- Let x be an arbitrary object of C. The induced map
$$(1_x)^* : \mathsf{C}(x,c) \longrightarrow \mathsf{C}(x,c)$$
is the identity map of the set $\mathsf{C}(x,c)$ since
$$(1_x)^*(h) = h \cdot 1_x = h = 1_{\mathsf{C}(x,c)}(h)$$
for every $h \in \mathsf{C}(x,c)$.
- Let $f : x \to y$ and $g : y \to z$ be two composable morphisms in C. We have the sequence of equalities
$$(gf)^*(h) = h(gf) = (hg)f = f^*(hg) = f^*(g^*(h))$$
for every $h \in \mathsf{C}(z,c)$, and therefore the equality $(gf)^* = f^* g^*$.

We have altogether shown that $\mathsf{C}(-,c)$ is a contravariant functor from C to Set.

Exercise 1.3.v

A functor from C to D is “the same” as a functor from C^{op} to D^{op}. Consequently, a functor from C^{op} to D is “the same” as a functor from $C^{opop} = C$ to D^{op}.

Exercise 1.3.vi

Definition of the composition of morphisms in $F \downarrow G$

Let

$$(h, k) : (d, e, f) \longrightarrow (d', e', f'), \quad (h', k') : (d', e', f') \longrightarrow (d'', e'', f'')$$

be two morphisms in $F \downarrow G$ that ought to be composable. We have by assumptions on the pairs (h, k) and (h', k') the following two commutative square diagrams:

$$\begin{array}{ccc} F(d) & \xrightarrow{\quad f \quad} & G(e) \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle G(k)} \\ F(d') & \xrightarrow{\quad f' \quad} & G(e') \end{array} \qquad \begin{array}{ccc} F(d') & \xrightarrow{\quad f' \quad} & G(e') \\ {\scriptstyle F(h')}\downarrow & & \downarrow{\scriptstyle G(k')} \\ F(d'') & \xrightarrow{\quad f'' \quad} & G(e'') \end{array}$$

By combining these two diagrams, with the first diagram above the second, we arrive at the following commutative diagram:

$$\begin{array}{ccc} F(d) & \xrightarrow{\quad f \quad} & G(e) \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle G(k)} \\ F(d') & \xrightarrow{\quad f' \quad} & G(e') \\ {\scriptstyle F(h')}\downarrow & & \downarrow{\scriptstyle G(k')} \\ F(d'') & \xrightarrow{\quad f'' \quad} & G(e'') \end{array}$$

By leaving out the center row of this diagram we arrive at the following commutative square diagram:

$$\begin{array}{ccc} F(d) & \xrightarrow{\quad f \quad} & G(e) \\ {\scriptstyle F(h')F(h)}\Big\downarrow & & \Big\downarrow{\scriptstyle G(k')G(k)} \\ F(d'') & \xrightarrow{\quad f'' \quad} & G(e'') \end{array}$$

This diagram can equivalently be rewritten as follows:

$$\begin{array}{ccc} F(d) & \xrightarrow{\quad f \quad} & G(e) \\ {\scriptstyle F(h'h)}\Big\downarrow & & \Big\downarrow{\scriptstyle G(k'k)} \\ F(d'') & \xrightarrow{\quad f'' \quad} & G(e'') \end{array}$$

The commutativity of this square diagram tells us that the pair $(h'h, k'k)$ is a morphism from (d, e, f) to (d'', e'', f'') in $F \downarrow G$. We define the composite $(h', k') \cdot (h, k)$ as $(h'h, k'k)$. In other words, the composition of morphisms in $F \downarrow G$ is componentwise.

The associativity of the composition of morphisms in the proposed category $F \downarrow G$ follows componentwise from the associativity of the compositions of morphisms in D and E.

It remains to prove the existence of identity morphisms in $F \downarrow G$.

We have for every object (d, e, f) in $F \downarrow G$ the following commutative square diagram:

$$\begin{array}{ccc} F(d) & \xrightarrow{\quad f \quad} & G(e) \\ {\scriptstyle 1_{F(d)}}\Big\downarrow & & \Big\downarrow{\scriptstyle 1_{G(e)}} \\ F(d) & \xrightarrow{\quad f \quad} & G(e) \end{array}$$

This diagram can equivalently be rewritten as follows:

$$\begin{array}{ccc} F(d) & \xrightarrow{\quad f \quad} & G(e) \\ {\scriptstyle F(1_d)}\Big\downarrow & & \Big\downarrow{\scriptstyle G(1_e)} \\ F(d) & \xrightarrow{\quad f \quad} & G(e) \end{array}$$

The commutativity of this diagram tells us that the pair $(1_d, 1_e)$ is a morphism from (d, e, f) to (d, e, f) in $F \downarrow G$. We have for every morphism

$$(h, k) : \ (d, e, f) \longrightarrow (d', e', f')$$

in $F \downarrow G$ the two sequences of equalities

$$(1_{d'}, 1_{e'}) \cdot (h, k) = (1_{d'} \cdot h, 1_{e'} \cdot k) = (h, k)$$

and

$$(h, k) \cdot (1_d, 1_e) = (h \cdot 1_d, k \cdot 1_e) = (h, k) .$$

This tells us that for every object (d, e, f) of $F \downarrow G$ the endomorphism $(1_d, 1_e)$ of (d, e, f) serves as the identity morphism of (d, e, f).

We have altogether constructed a category $F \downarrow G$.

The functors dom and codom

We define the "domain functor" $\mathrm{dom} : \ F \downarrow G \to \mathsf{D}$ as $\mathrm{dom}((d, e, f)) = d$ on objects and as $\mathrm{dom}((h, k)) = h$ on morphisms. Similarly, we define the "codomain functor" $\mathrm{codom} : \ F \downarrow G \to \mathsf{E}$ as $\mathrm{codom}((d, e, f)) = e$ on objects and as $\mathrm{codom}((h, k)) = k$ on morphisms. The actions of dom and codom can more graphically be depicted as follows:

$$\begin{array}{ccccccc} d & & F(d) & \xrightarrow{f} & G(e) & & e \\ {\scriptstyle h}\big\downarrow & \leftsquigarrow & {\scriptstyle F(h)}\big\downarrow & & \big\downarrow{\scriptstyle G(k)} & \rightsquigarrow & \big\downarrow{\scriptstyle k} \\ d' & & F(d') & \xrightarrow{f'} & G(e') & & e' \end{array}$$

The assignments dom and codom are indeed functors because identities in $F \downarrow G$ and composition of morphisms in $F \downarrow G$ work componentwise.

Exercise 1.3.vii

Let C be a category and let c be an arbitrary object of C. Let $\mathbb{1}$ be the singleton category consisting of a single object $*$ and only the morphism 1_*. Let F be the constant functor from $\mathbb{1}$ to C corresponding to the object c, i.e., the functor

$$F : \ \mathbb{1} \longrightarrow \mathsf{C} , \quad * \longmapsto c , \quad 1_* \longmapsto 1_c .$$

The comma category $1_{\mathsf{C}} \downarrow F$ looks as follows:

- The objects of $1_{\mathsf{C}} \downarrow F$ are triples $(x, *, f)$ consisting of an object x of C and a morphism $f:\ 1_{\mathsf{C}}(x) \to F(*)$, i.e., a morphism $f:\ x \to c$.
- A morphism from $(x, *, f)$ to $(y, *, g)$ is a pair $(\varphi, 1_*)$ consisting of a morphism $\varphi:\ x \to y$ in C such that the following diagram commutes:

$$\begin{array}{ccc} 1_{\mathsf{C}}(x) & \xrightarrow{\quad f \quad} & F(*) \\ {\scriptstyle 1_{\mathsf{C}}(\varphi)}\Big\downarrow & & \Big\downarrow{\scriptstyle F(1_*)} \\ 1_{\mathsf{C}}(y) & \xrightarrow[\quad g \quad]{} & F(*) \end{array}$$

This diagram can be simplified as follows:

$$\begin{array}{ccc} x & \xrightarrow{\quad f \quad} & c \\ {\scriptstyle \varphi}\Big\downarrow & & \Big\downarrow{\scriptstyle 1_c} \\ y & \xrightarrow[\quad g \quad]{} & c \end{array}$$

A further simplification yields the following diagram:

$$\begin{array}{ccc} x & \xrightarrow{\qquad \varphi \qquad} & y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & c & \end{array}$$

- The composite of two morphisms

$$(\varphi, 1_*):\ (x, *, f) \longrightarrow (y, *, g)\,, \quad (\psi, 1_*):\ (y, *, g) \longrightarrow (z, *, h)$$

is given by

$$(\psi, 1_*) \cdot (\varphi, 1_*) = (\psi\varphi, 1_* 1_*) = (\psi\varphi, 1_*)\,.$$

We find that the comma category $1_{\mathsf{C}} \downarrow F$ is isomorphic to the slice category C/c via

$$(x, *, f) \longmapsto (x, f)\,, \quad (\varphi, 1_*) \longmapsto \varphi\,.$$

We can show similarly that the slice category c/C is isomorphic to the comma category $F \downarrow 1_{\mathsf{C}}$.

The domain functor $(1_{\mathsf{C}} \downarrow F) \to \mathsf{C}$ corresponds to the functor $\mathsf{C}/c \to \mathsf{C}$ given by

$$\left(\begin{array}{c} x \\ \big\downarrow \\ c \end{array} \right) \longmapsto x \,, \qquad \left(\begin{array}{ccc} x & \xrightarrow{\quad\varphi\quad} & y \\ & \searrow \quad \swarrow & \\ & c & \end{array} \right) \longmapsto \left(x \xrightarrow{\varphi} y \right).$$

The codomain functor $(1_{\mathsf{C}} \downarrow F) \to \mathbb{1}$ corresponds to the constant functor $\mathsf{C}/c \to \mathbb{1}$.

The domain functor $(F \downarrow 1_{\mathsf{C}}) \to \mathbb{1}$ corresponds to the constant functor $c/\mathsf{C} \to \mathbb{1}$. The codomain functor $(F \downarrow 1_{\mathsf{C}}) \to \mathsf{C}$ corresponds to the functor $c/\mathsf{C} \to \mathsf{C}$ given by

$$\left(\begin{array}{c} c \\ \big\downarrow \\ x \end{array} \right) \longmapsto x \,, \qquad \left(\begin{array}{ccc} & c & \\ & \swarrow \quad \searrow & \\ x & \xrightarrow{\quad\varphi\quad} & y \end{array} \right) \longmapsto \left(x \xrightarrow{\varphi} y \right).$$

Exercise 1.3.viii

Let $\mathbb{1}$ be the category consisting of a single object $*$ and only the respective identity morphism. Let $\mathbb{2}$ be the category consisting of two objects, named 0 and 1, and a single non-identity morphism, namely $f \colon 0 \to 1$. The unique functor from $\mathbb{2}$ to $\mathbb{1}$ maps f onto the identity morphism of $*$, which is an isomorphism. But f is not an isomorphism in $\mathbb{2}$ because there exists no morphism from 1 to 0 in $\mathbb{2}$.

Exercise 1.3.ix

For isomorphisms

Every isomorphism of groups $\varphi \colon G \to H$ induces isomorphisms of groups

$$\begin{aligned} \mathrm{Z}(\varphi) \colon \; \mathrm{Z}(G) &\longrightarrow \mathrm{Z}(H) \,, & g &\longmapsto \varphi(g) \,, \\ \mathrm{C}(\varphi) \colon \; \mathrm{C}(G) &\longrightarrow \mathrm{C}(H) \,, & g &\longmapsto \varphi(g) \,, \\ \mathrm{Aut}(\varphi) \colon \; \mathrm{Aut}(G) &\longrightarrow \mathrm{Aut}(H) \,, & \psi &\longmapsto \varphi \psi \varphi^{-1} \,. \end{aligned}$$

In this way, the constructions Z, C and Aut become functors from $\mathsf{Group}_{\mathrm{iso}}$ to Group.

For epimorphisms

We use in following without proof that a homomorphism of groups is an epimorphism in Group if and only if it is surjective.

- Let $\varphi \colon G \to H$ be an epimorphism of groups. We claim that

$$\varphi(\mathrm{Z}(G)) \subseteq \mathrm{Z}(H)\,.$$

  Indeed, let z be an element of $\mathrm{Z}(G)$ and let h be an element of H. There exists by assumption some element g of G with $h = \varphi(g)$. It follows that

$$\varphi(z)h = \varphi(z)\varphi(g) = \varphi(zg) = \varphi(gz) = \varphi(g)\varphi(z) = h\varphi(z)\,.$$

  This shows that $\varphi(z)$ is contained in $\mathrm{Z}(H)$, as claimed.

  It follows that the homomorphism φ restricts to a homomorphism of groups from $\mathrm{Z}(G)$ to $\mathrm{Z}(H)$. This observation allows us to extend Z to a functor from $\mathrm{Group}_{\mathrm{epi}}$ to Group.

- Every homomorphism of groups $\varphi \colon G \to H$ induces a homomorphism groups from $\mathrm{C}(G)$ to $\mathrm{C}(H)$ by restriction. This entails that C extends to a functor from $\mathrm{Group}_{\mathrm{epi}}$ to Group.

- An epimorphism of groups $\varphi \colon G \to H$ does not necessarily induce a map from $\mathrm{Aut}(G)$ to $\mathrm{Aut}(H)$: an automorphism ψ of G descends to an endomorphism of H if and only if $\psi(\ker(\varphi)) \subseteq \ker(\varphi)$.

  There nevertheless exists a functor from $\mathrm{Group}_{\mathrm{epi}}$ to Group that assigns to each group its automorphism group. This is due to the following observation:

  Claim 1 ([MSE18b]). Every functor from $\mathrm{Group}_{\mathrm{iso}}$ to Group can be extended to a functor from $\mathrm{Group}_{\mathrm{epi}}$ to Group.

  This observation in turn relies on the following observation:

  Claim 2. Let $\varphi \colon G \to H$ and $\psi \colon H \to K$ be two epimorphisms of groups. The composite $\psi\varphi$ is an isomorphism if and only if both φ and ψ are isomorphisms.

  Proof. If both φ and ψ are isomorphisms then their composite $\psi\varphi$ is again an isomorphism.

Suppose conversely that $\psi\varphi$ is an isomorphism. This entails that $\psi\varphi$ is a monomorphism, whence φ is a monomorphism. As φ is both a monomorphism and an epimorphism in Group, it is an isomorphism. It follows that $\psi = \psi\varphi \cdot \varphi^{-1}$ is a composite of isomorphisms and therefore also an isomorphism. ∎

Proof of Claim 1. Let F be a functor from $\mathrm{Group}_{\mathrm{iso}}$ to Group. We define an extension F' of F by letting $F'(\varphi)$ be the trivial homomorphism from $F(G)$ to $F(H)$ for every epimorphism $\varphi : G \to H$ that is not an isomorphism. We need to prove that the assignment F' is functorial. More specifically, we need to check that F' is compatible with both identities and composition.

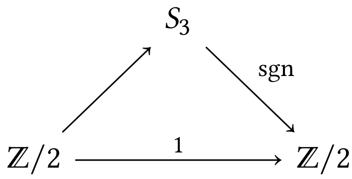
- Let G be any group. The identity homomorphism 1_G is an isomorphism, so we have
$$F'(1_G) = F(1_G) = 1_{F(G)} = 1_{F'(G)}$$
by the functoriality of F.
- Let $\varphi : G \to H$ and $\psi : H \to K$ be two epimorphisms of groups. We need to show that $F'(\psi\varphi) = F'(\psi)F'(\varphi)$. We distinguish between two cases:

  Case 1. Suppose that both φ and ψ are isomorphisms. Then their composite $\psi\varphi$ is again an isomorphism, whence
  $$F'(\psi\varphi) = F(\psi\varphi) = F(\psi)F(\varphi) = F'(\psi)F'(\varphi)$$
  by the functoriality of F.

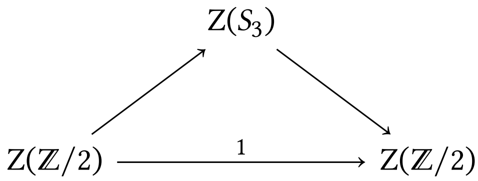
  Case 2. Suppose that either φ or ψ is not an isomorphism. By definition of F', either $F'(\varphi)$ or $F'(\psi)$ is trivial. The composite $F'(\psi)F'(\varphi)$ is therefore again trivial. It also follows from Claim 2 that $\psi\varphi$ is not an isomorphism, whence $F'(\psi\varphi)$ is trivial. This shows that $F'(\psi\varphi) = F'(\psi)F'(\varphi)$, as both sides are trivial with the same domain and same codomain. ∎

For homomorphisms

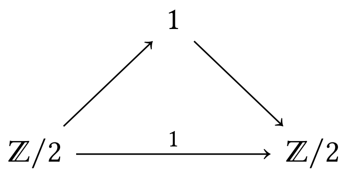
- If Z were to extend to a functor from Group to Group, then the commutative diagram



would result in the following commutative diagram:



We have $\mathrm{Z}(\mathbb{Z}/2) = \mathbb{Z}/2$ and $\mathrm{Z}(S_3) = 1$ (the trivial group), and therefore would get the following commutative diagram:



The commutativity of this diagram would mean that the identity homomorphism of $\mathbb{Z}/2$ is trivial. But this is not the case!

- The construction C extends to a functor from Group to Group in the usual way: every homomorphism of groups $\varphi : G \to H$ satisfies $\varphi(\mathrm{C}(G)) \subseteq \mathrm{C}(H)$, and therefore restricts to a homomorphism $\mathrm{C}(\varphi)$ from $\mathrm{C}(G)$ to $\mathrm{C}(H)$.
- It can be shown that Aut cannot be extended to a functor from Group to Group. We refer to [MSE15] for more details on this claim.

Exercise 1.3.x

Let $\varphi : G \to H$ be a homomorphism of groups. For every element g of G, the homomorphism φ maps the conjugacy class of g into the conjugacy class of $\varphi(g)$. The homomorphism φ does therefore induce a map

$$\mathrm{Conj}(\varphi) : \mathrm{Conj}(G) \longrightarrow \mathrm{Conj}(H), \quad [g] \longmapsto [\varphi(g)].$$

We need to verify the functoriality of Conj:

- We have for any two homomorphism of groups $\varphi : G \to H$ and $\psi : H \to K$

the sequence of equalities

$$\begin{aligned} \mathrm{Conj}(\psi)(\mathrm{Conj}(\varphi)([g])) &= \mathrm{Conj}(\psi)([\varphi(g)]) \\ &= [\psi(\varphi(g))] \\ &= [(\psi\varphi)(g)] \\ &= \mathrm{Conj}(\psi\varphi)([g]) \end{aligned}$$

for every $g \in G$, and therefore the equality

$$\mathrm{Conj}(\psi\varphi) = \mathrm{Conj}(\psi)\,\mathrm{Conj}(\varphi)\,.$$

- We also have for every group G the sequence of equalities

$$\mathrm{Conj}(1_G)([g]) = [1_G(g)] = [g] = 1_{\mathrm{Conj}(G)}([g])$$

for every $g \in G$, and therefore the equality

$$\mathrm{Conj}(1_G) = 1_{\mathrm{Conj}(G)}\,.$$

This proves the functoriality of Conj.

1.4 Naturality

Example 1.4.3

(vii)

Suppose that α is a natural transformation from F to G, where F is the identity functor of $\mathrm{Vect}_{\Bbbk}$, and

$$G\colon\ \mathrm{Vect}_{\Bbbk} \longrightarrow \mathrm{Vect}_{\Bbbk}\,, \quad V \longmapsto V \otimes V\,, \quad f \longmapsto f \otimes f\,.$$

We then have for every linear map $f\colon \Bbbk \to V$ the following commutative square diagram:

$$\begin{array}{ccc} \Bbbk & \xrightarrow{\alpha_{\Bbbk}} & \Bbbk \otimes \Bbbk \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f \otimes f} \\ V & \xrightarrow{\alpha_V} & V \otimes V \end{array}$$

The linear map α is given by $\alpha_{\Bbbk}(1) = \lambda 1 \otimes 1$ for some scalar λ in $\Bbbk$ because both $\Bbbk$ and $\Bbbk \otimes \Bbbk$ are one-dimensional with basis elements 1 and $1 \otimes 1$ respectively. Let v be an arbitrary vector of V and let f be the unique linear map from $\Bbbk$ to V with $f(1) = v$. It follows from the commutativity of the above square diagram that

$$\begin{aligned} \alpha_V(v) &= \alpha_V(f(1)) \\ &= (f \otimes f)(\alpha_{\Bbbk}(1)) \\ &= (f \otimes f)(\lambda 1 \otimes 1) \\ &= \lambda (f \otimes f)(1 \otimes 1) \\ &= \lambda f(1) \otimes f(1) \\ &= \lambda v \otimes v \end{aligned}$$

for every $v \in V$. This shows that the linear map α_V is uniquely determined by the scalar λ via $\alpha_V(v) = \lambda v \otimes v$ for every $v \in V$.

However, the map

$$V \longrightarrow V \otimes V\,, \quad v \longmapsto \lambda v \otimes v$$

is linear if and only if either $V = 0$, or $\lambda = 0$, or if simultaneously $\dim(V) = 1$ and $\Bbbk = \mathbb{F}_2$. This can be seen as follows:

- Suppose that V is at least two-dimensional. There then exist two linearly independent vectors v_1 and v_2 in V. The vectors

  $$(v_1 + v_2) \otimes (v_1 + v_2) = v_1 \otimes v_1 + v_1 \otimes v_2 + v_2 \otimes v_1 + v_2 \otimes v_2$$

  and $v_1 \otimes v_1 + v_2 \otimes v_2$ are then distinct. The map $v \mapsto \lambda v \otimes v$ is therefore not additive if λ is nonzero.

- Suppose that $\Bbbk$ is not $\mathbb{F}_2$. There then exists a scalar μ in $\Bbbk$ that is distinct to 0 and 1, and therefore satisfies $\mu^2 \neq \mu$. If V is also nonzero, then there exists a nonzero vector v in V. The two vectors $(\mu v) \otimes (\mu v) = \mu^2 v \otimes v$ and $\mu v \otimes v$ are then distinct. The map $v \mapsto \lambda v \otimes v$ is therefore not homogeneous if λ is nonzero.

This shows that the only natural transformation from F to G is given by the zero map in each of its coordinates (corresponding to the above case $\lambda = 0$).

Exercise 1.4.i

Let C be the domain of the two functors F and G.

The naturality of α ensures for every morphism $f : x \to y$ in C the commutativity of the following square diagram:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ G(x) & \xrightarrow{G(f)} & G(y) \end{array}$$

This commutativity is equivalent to the equality $\alpha_y \cdot F(f) = G(f) \cdot \alpha_x$. We can rearrange this equality to $F(f) \cdot \alpha_x^{-1} = \alpha_y^{-1} \cdot G(f)$. This new equality gives us the commutativity of the following diagram:

$$\begin{array}{ccc} G(x) & \xrightarrow{G(f)} & G(y) \\ {\scriptstyle \alpha_x^{-1}}\downarrow & & \downarrow{\scriptstyle \alpha_y^{-1}} \\ F(x) & \xrightarrow{F(f)} & F(y) \end{array}$$

The commutativity of this diagram, for every morphism $f : x \to y$ in C, tells us that the family $(\alpha_x^{-1})_{x \in \mathsf{C}}$ is a natural transformation from G to F.

Exercise 1.4.ii

Let G and H be two groups. We have already seen in Exercise 1.3.i (page 22 of the textbook) that a functor $\Phi : \mathrm{B}G \to \mathrm{B}H$ is the same a homomorphism of groups $\varphi : G \to H$, via the assignments $\Phi(*_{\mathrm{B}G}) = *_{\mathrm{B}H}$ and $\Phi(g) = \varphi(g)$ for every $g \in G$.

Let $\Phi, \Psi : \mathrm{B}G \to \mathrm{B}H$ be two functors with corresponding homomorphisms of groups $\varphi, \psi : G \to H$. A natural transformation $\alpha : \Phi \Rightarrow \Psi$ is a family $(\alpha_x)_{x \in \mathrm{B}G}$ consisting of morphisms $\alpha_x : \Phi(x) \to \Psi(x)$ for $x \in \mathrm{B}G$ such that the square diagram

$$\begin{array}{ccc} \Phi(x) & \xrightarrow{\Phi(g)} & \Phi(y) \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ \Psi(x) & \xrightarrow{\Psi(g)} & \Psi(y) \end{array}$$

commutes for every morphism $g : x \to y$ in BG. Given the specific shapes of BG and BH, the entire transformation α consists of the single element α_* of H, which has to satisfy the commutativity of the diagram

$$\begin{array}{ccc} * & \xrightarrow{\varphi(g)} & * \\ {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ * & \xrightarrow{\psi(g)} & * \end{array}$$

for every $g \in G$. A natural transformation from Φ to Ψ is therefore "the same" as an element h of H such that $h\varphi(g) = \psi(g)h$ for every $g \in G$. In other words, the element h needs to satisfy $\psi(g) = h\varphi(g)h^{-1}$ for every $g \in G$.

We see overall that there exists a natural transformation from Φ to Ψ if and only if φ is conjugated to ψ. More explicitly, natural transformations from Φ to Ψ correspond bijectively to elements of H that realize the conjugation of φ and ψ.

Exercise 1.4.iii

Let P and Q be two preordered sets with corresponding categories P and Q.

We have seen in Exercise 1.3.ii that a functor F from P to Q is "the same" as an isotone map f from P to Q, via the assignments $F(p) = f(p)$ for every $p \in P$, and $F(p_1 \to p_2) = (f(p_1) \to f(p_2))$ for all $p_1, p_2 \in P$. Let in the following $F, G :$ P $\to$ Q be two functors with corresponding isotone maps $f, g : P \to Q$.

There exists at most one natural transformation from F to G since for every element p on P there exists at most one morphism from $F(p)$ to $G(p)$ in Q. The existence of a transformation from F to G is equivalent to the existence of a family $(\alpha_p)_{p \in P}$ of morphisms $\alpha_p : f(p) \to g(p)$, which in turn is equivalent to having $f(p) \leq g(p)$ for every $p \in P$. Such a transformation is then automatically natural since every diagram in Q commutes (because every two morphisms in Q with the same domain and the same codomain are already equal).

We find overall that there exists a natural transformation from F to G if and only if $f \leq g$, in the sense that $f(p) \leq g(p)$ for every $p \in P$, and that this natural transformation is then unique.

Exercise 1.4.iv

If $f_* = g_*$, then by considering more specifically $f_*, g_* \colon \mathrm{C}(c, c) \to \mathrm{C}(c, d)$ we find that

$$f = f \cdot 1_c = f_*(1_c) = g_*(1_c) = g \cdot 1_c = g \,.$$

Similarly, if $f^* = g^*$, then by considering $f^*, g^* \colon \mathrm{C}(d, d) \to \mathrm{C}(c, d)$ we find that

$$f = 1_d \cdot f = f^*(1_d) = g^*(1_d) = 1_d \cdot g = g \,.$$

(One could also derive the implication $f^* = g^* \implies f = g$ from the implication $f_* = g_* \implies f = g$ via duality.)

Exercise 1.4.v

For every object $x = (d, e, f)$ of the comma category $F \downarrow G$ let α_x be the morphism

$$f \colon Fd \longrightarrow Ge \,.$$

Then, since $d = \operatorname{dom} x$ and $e = \operatorname{codom} x$, the component α_x is a morphism from $F \operatorname{dom} x$ to $G \operatorname{codom} x$. Therefore, α is a transformation from $F \operatorname{dom}$ to $G \operatorname{codom}$. We claim that this transformation is natural.

Indeed, let $(h, k) \colon x \to x'$ be an arbitrary morphism in $F \downarrow G$ with domain $x = (d, e, f)$ and codomain $x' = (d', e', f')$. This means that we have the following commutative diagram:

$$\begin{array}{ccc} Fd & \xrightarrow{\;f\;} & Ge \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle Gk} \\ Fd' & \xrightarrow{\;f'\;} & Ge' \end{array}$$

This diagram can be rewritten as follows:

$$\begin{array}{ccc} F\operatorname{dom} x & \xrightarrow{\alpha_x} & G\operatorname{codom} x \\ {\scriptstyle F\operatorname{dom}(h,k)}\downarrow & & \downarrow{\scriptstyle G\operatorname{codom}(h,k)} \\ F\operatorname{dom} x' & \xrightarrow{\alpha_{x'}} & G\operatorname{codom} x' \end{array}$$

The commutativity of this square diagram shows precisely that α is natural.

Exercise 1.4.vi

In the given diagrams we have morphisms between objects in the codomain category of F and objects in the codomain category of G. For this to make sense we need both functors to have the same codomain category.

1.5 Equivalence of categories

Equivalences between skeletal categories are isomorphisms

For every category C let $\operatorname{Iso}(\mathsf{C})$ be the class of isomorphism classes of objects of C. We note that every functor $F \colon \mathsf{C} \to \mathsf{D}$ induces a function $\operatorname{Iso}(F)$ from $\operatorname{Iso}(\mathsf{C})$ to $\operatorname{Iso}(\mathsf{D})$ via $[c] \mapsto [Fc]$. The construction Iso is functorial.

Lemma 1.A. Let $F, G \colon \mathsf{C} \to \mathsf{D}$ be two parallel functors. If F and G are isomorphic, then the induced maps $\operatorname{Iso}(F)$ and $\operatorname{Iso}(G)$ are equal.

Proof. We have for every object c of C the isomorphism $Fc \cong Gc$ and therefore the equality

$$\operatorname{Iso}(F)(c) = [Fc] = [Gc] = \operatorname{Iso}(G)(c)\,.$$

This shows that $\operatorname{Iso}(F) = \operatorname{Iso}(G)$. ∎

Proposition 1.B. Let $F \colon \mathsf{C} \to \mathsf{D}$ be an equivalence of categories with quasi-inverse $G \colon \mathsf{D} \to \mathsf{C}$. The induced map $\operatorname{Iso}(F)$ is bijective with inverse $\operatorname{Iso}(G)$.

Proof. There exists an isomorphism of functors between GF and 1_{C}. It follows from Lemma 1.A that

$$\operatorname{Iso}(G)\operatorname{Iso}(F) = \operatorname{Iso}(GF) = \operatorname{Iso}(1_{\mathsf{C}}) = 1_{\operatorname{Iso}(\mathsf{C})}\,.$$

We find in the same way that also $\operatorname{Iso}(F)\operatorname{Iso}(G) = 1_{\operatorname{Iso}(\mathsf{D})}$. ∎

Proposition 1.C. Let F be a functor that is full, faithful, and induces a bijection between objects. Then F is an isomorphism. ∎

Proof. It follows that F also induces a bijection on morphisms. ∎

Corollary 1.D. An equivalence between skeletal categories is already an isomorphism of categories.

Proof. Let $F : \mathsf{C} \to \mathsf{D}$ be an equivalence between skeletal categories. It follows from Proposition 1.B and C and D being skeletal that F induces a bijection between objects. The functor F is also full and faithful. It is therefore an isomorphism of categories by Proposition 1.C. ∎

A category equivalent to a groupoid is itself a groupoid

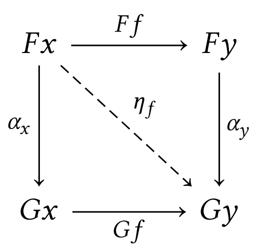
This statement is a consequence of Exercise 1.5.iv, as equivalences reflect isomorphisms.

Exercise 1.5.i

We first prove an auxiliary result.

Proposition 1.E. Let C and D be categories and let $F, G : \mathsf{C} \to \mathsf{D}$ be functors.

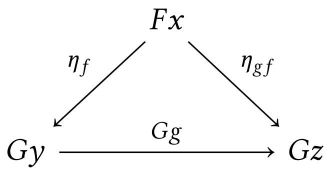
1. Let $\alpha : F \Rightarrow G$ be a natural transformation. For every morphism $f : x \to y$ in C let $\eta_f : Fx \to Gy$ be the diagonal morphism in the resulting commutative square diagram:

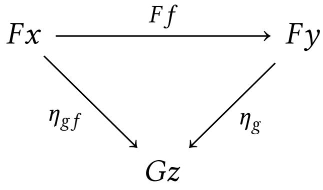


(1.2)

   In other words, $\eta_f = \alpha_y \cdot Ff$ and also $\eta_f = Gf \cdot \alpha_x$.

Then the following triangular diagrams commute for all composable morphisms $f\colon x \to y$ and $g\colon y \to z$ in C:





In other words, we have

$$\eta_{gf} = Gg \cdot \eta_f \quad \text{and} \quad \eta_{gf} = \eta_g \cdot Ff\,. \tag{1.3}$$

2. Suppose conversely that $(\eta_f)_f$ is a family of morphisms $\eta_f\colon Fx \to Gy$, where $f\colon x \to y$ ranges through the morphisms in C, such that the conditions (1.3) hold for all morphisms $f\colon x \to y$ and $g\colon y \to z$ in C.

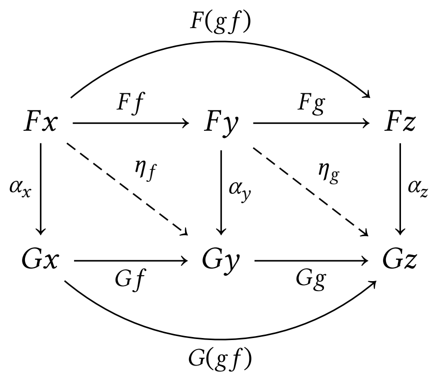
   Then there exists a unique natural transformation $\alpha\colon F \Rightarrow G$ such that the diagram (1.2) commutes for every morphism $f\colon x \to y$ in C.

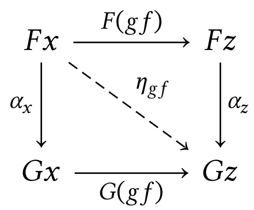
3. The above two constructions are mutually inverse, and thus result in a bijection between

   - natural transformations $\alpha\colon F \Rightarrow G$ and
   - families $(\eta_f)_f$ of morphisms $\eta_f\colon Fx \to Gy$, where $f\colon x \to y$ ranges through the morphisms in C, such that $\eta_{gf} = Gg \cdot \eta_f$ and $\eta_{gf} = \eta_g \cdot Ff$ for all morphisms $f\colon x \to y$ and $g\colon y \to z$ in C.

Proof.

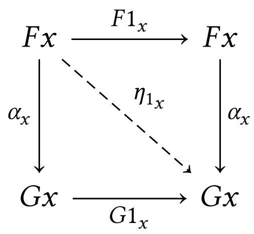
1. We have for all morphisms $f\colon x \to y$ and $g\colon y \to z$ in C the following two commutative diagrams:





The outer square part of these diagrams is the same. The overall diagonal morphism from Fx to Gz is therefore the same in both diagrams. This entails that $\eta_{gf} = \eta_g \cdot Ff$ as well as $\eta_{gf} = Gg \cdot \eta_f$.

2. If such a natural transformation α were to exist, then we would have for every object x of C the following commutative square diagram:



It would then follow that

$$\eta_{1_x} = \alpha_x \cdot F1_x = \alpha_x \cdot 1_{Fx} = \alpha_x \,.$$

This shows the uniqueness of α.

To prove the existence of α we conversely set

$$\alpha_x \coloneqq \eta_{1_x}$$

for every object x of C, which is a morphism from Fx to Gx. We need to show that the family $\alpha \coloneqq (\alpha_x)_x$ is a natural transformation from F to G, and that $\eta_f = \alpha_y \cdot Ff$ and $\eta_f = Gf \cdot \alpha_x$ for every morphism $f \colon x \to y$ in C. These two equalities hold because

$$\eta_f = \eta_{1_y \cdot f} = \eta_{1_y} \cdot Ff = \alpha_y \cdot Ff$$

and similarly

$$\eta_f = \eta_{f \cdot 1_x} = Gf \cdot \eta_{1_x} = Gf \cdot \alpha_x \,,$$

and the resulting sequence of equalities $\alpha_y \cdot Ff = \eta_f = Gf \cdot \alpha_x$ also shows that the family α is natural.

3. Let $\alpha \colon F \Rightarrow G$ be a natural transformation. Let $(\eta_f)_f$ be the resulting family of morphisms $\eta_f \colon Fx \to Gy$, where $f \colon x \to y$ ranges through the morphisms in C, given by $\eta_f = \alpha_y \cdot Ff$ (and also $\eta_f = Gf \cdot \alpha_x$). Let α' be the resulting natural transformation from F to G given by $\alpha'_x = \eta_{1_x}$ for

every object x of C. We then have for every object x of C the sequence of equalities

$$\alpha'_x = \eta_{1_x} = \alpha_x \cdot F1_x = \alpha_x \cdot 1_{Fx} = \alpha_x \,,$$

which shows that $\alpha' = \alpha$.

Let now conversely $(\eta_f)_f$ be a family of morphisms $\eta_f : Fx \to Gy$, where $f : x \to y$ ranges through the morphisms in C, satisfying the two conditions $\eta_{gf} = Gg \cdot \eta_f$ and $\eta_{gf} = \eta_g \cdot Ff$ for all morphisms $f : x \to y$ and $g : y \to z$ in C. Let α be the resulting natural transformation from F to G given by $\alpha_x = \eta_{1_x}$ for every object x of C. Let $(\eta'_f)_f$ be the resulting family of morphisms $\eta'_f : Fx \to Gy$, where once again $f : x \to y$ ranges through the morphisms in C, with η'_f given by $\eta'_f = \alpha_y \cdot Ff$ (and also equivalently $\eta'_f = Gf \cdot \alpha_x$). Then

$$\eta'_f = \alpha_y \cdot Ff = \eta_{1_y} \cdot Ff = \eta_{1_y \cdot f} = \eta_f$$

for every morphism $f : x \to y$ in C. ∎

We now return to the exercise at hand. We note that a functor

$$H : \mathsf{C} \times \mathbb{2} \longrightarrow \mathsf{D}$$

consists of the following data:

D1. For every object x of C an object $H(x, 0)$ of D.

D2. For every object x of C an object $H(x, 1)$ of D.

D3. For every morphism $f : x \to y$ in C a morphism $H(f, 1_0)$ from $H(x, 0)$ to $H(y, 0)$ in D.

D4. For every morphism $f : x \to y$ in C a morphism $H(f, 1_1)$ from $H(x, 1)$ to $H(y, 1)$ in D.

D5. For every morphism $f : x \to y$ in C a morphism $H(f, j)$ from $H(x, 0)$ to $H(y, 1)$ in D, where $j : 0 \to 1$ is the unique non-identity morphism in $\mathbb{2}$.

These data are subject to the following conditions:

C1. For every object x of C the two equalities

C1.a. $H(1_x, 1_0) = 1_{H(x,0)}$ and

C1.b. $H(1_x, 1_1) = 1_{H(x,1)}$.

C2. For all morphisms $f : x \to y$ and $g : y \to z$ in C the four equalities

C2.a. $H(gf, 1_0) = H(g, 1_0) \cdot H(f, 1_0)$,

C2.b. $H(gf, 1_1) = H(g, 1_1) \cdot H(g, 1_0)$,

C2.c. $H(gf, j) = H(g, j) \cdot H(f, 1_0)$,

C2.d. $H(gf, j) = H(g, 1_1) \cdot H(f, j)$.

For every object x of C let

$$Fx := H(x, 0)\,, \quad Gx := H(x, 1)\,,$$

and for every morphism $f : x \to y$ in C let

$$Ff := H(f, 1_0)\,, \quad Gf := H(f, 1_1)\,, \quad \eta_f := H(f, j)\,.$$

The datum of F is equivalent to the data D1 and D3, the datum of G is equivalent to the data D2 and D4, and the datum of η is equivalent to the datum D5. The combination of conditions C1.a and C2.a is equivalent to the functoriality of F, the combination of conditions C1.b and C2.b is equivalent to the functoriality of G. The combination of conditions C2.c and C2.d is then equivalent to η defining a natural transformation from F to G via Proposition 1.E.

This shows that functors $H : \mathsf{C} \times \mathbb{2} \to \mathsf{D}$ correspond to pairs of functors $F, G : \mathsf{C} \to \mathsf{D}$ together with a natural transformation $\alpha : F \Rightarrow G$. This correspondence is given by $F = Hi_0$ and $G = Hi_1$, and the natural transformation α corresponds to the morphisms $H(f, j) : Fx \to Gy$, where $f : x \to y$ ranges through the morphisms in C, as laid out in Proposition 1.E.

Exercise 1.5.ii

We will show that the category Γ^{op} is isomorphic to Fin^∂. As Fin^∂ is equivalent to Fin_*, this then also shows that Γ^{op} is equivalent to Fin_*. (We have seen in Example 1.5.6 that Set^∂ is equivalent to Set_*. This equivalence restricts to an equivalence of categories between Fin^∂ and Fin_*.)

To prove the claimed isomorphism between Γ^{op} and Fin^∂, we will construct mutually inverse contravariant functors F and G between Γ and Fin^∂. The main observation is that a partially defined function $f : T \to S$ is uniquely determined by its fibres $f^{-1}(s)$, which are disjoint subsets of T indexed by S. Such an indexed family of pairwise disjoint subsets has the same data as a morphism from S to T in Γ.

The functor F

We start with the functor $F\colon \Gamma^{\mathrm{op}} \to \mathsf{Fin}^{\partial}$.

We define F on objects as $F(S) \coloneqq S$ for every finite set S.

For every morphism $\theta\colon S \to T$ in Γ let $F(\theta)$ be the partially defined function from T to S given by

$$F(\theta)(t) = s \quad \text{if and only if} \quad t \in \theta(s)$$

for all $s \in S$, $t \in T$. In other words, the set $\theta(s)$ is the fibre of s under $F(\theta)$:

$$F(\theta)^{-1}(s) = \theta(s)\,.$$

The partially defined function $F(\theta)$ is well-defined since the sets $\theta(s)$, where s ranges through S, are pairwise disjoint.[2]

We us now verify the functoriality of F.

Let S be a finite set, regarded as an object of Γ, and let ι_S be the identity morphism of S in Γ. In other words, ι_S is the function

$$\iota_S\colon S \longrightarrow \mathrm{P}(S)\,, \quad s \longmapsto \{s\}\,.$$

The resulting function $F(\iota_S)\colon S \to S$ is then given by $F(\iota_S)(s) = s$ for every $s \in S$, whence $F(\iota_S)$ is the identity function on S. This shows that the assignment F preserves identities.

Let θ be a morphism in Γ from a set S to a set T, and let σ be a morphism in Γ from T to a set U. (Therefore, θ and σ are functions $\theta\colon S \to \mathrm{P}(T)$ and $\sigma\colon T \to \mathrm{P}(U)$.) We have for all $s \in S$, $u \in U$ the sequence of equivalences

$$\begin{aligned}
& F(\sigma\theta)(u) = s \\
\iff\ & u \in (\sigma\theta)(s) \\
\iff\ & u \in \textstyle\bigcup_{t \in \theta(s)} \sigma(t) \\
\iff\ & \text{there exists some } t \in \theta(s) \text{ with } u \in \sigma(t) \\
\iff\ & \text{there exists some } t \in T \text{ with } t \in \theta(s) \text{ and } u \in \sigma(t) \\
\iff\ & \text{there exists some } t \in T \text{ with } F(\theta)(t) = s \text{ and } F(\sigma)(u) = t \\
\iff\ & F(\theta)(F(\sigma)(u)) = s \\
\iff\ & (F(\theta) \circ F(\sigma))(u) = s\,.
\end{aligned}$$

This shows that $F(\sigma\theta) = F(\theta) \circ F(\sigma)$.

We have thus shown that F is a contravariant functor from Γ to Fin^{∂}.

[2] The partially defined function $F(\theta)$ is a total function if and only if the set T is completely covered by the sets $\theta(s)$ with $s \in S$.

The functor G

We now define the functor G.

The action of G on objects is given by $G(S) = S$ for every finite set S.

For every partially defined function $f : S \to T$ between finite sets S and T let $G(f)$ be the induced function

$$G(f) : T \longrightarrow \mathrm{P}(S) , \quad t \longmapsto f^{-1}(t) .$$

The fibres $f^{-1}(t)$, where t ranges through T, are pairwise disjoint. The function $G(f)$ is therefore a morphism from T to S in Γ.

We have to verify the contravariant functoriality of G.

Let S be a set and let 1_S be the identity function on S. Then

$$G(1_S)(s) = 1_S^{-1}(s) = \{s\} = \iota_S(s)$$

for every $s \in S$, where ι_S denotes the identity morphism of S in Γ, and therefore $G(1_S) = \iota_S$. This shows that the assignment G preserves identities.

Let $f : S \to T$ and $g : T \to U$ be partially defined functions between sets S, T and U. We have for every $u \in U$ the sequence of equalities

$$\begin{aligned} G(g \circ f)(u) &= (g \circ f)^{-1}(u) \\ &= f^{-1}(g^{-1}(u)) \\ &= \textstyle\bigcup_{t \in g^{-1}(u)} f^{-1}(t) \\ &= \textstyle\bigcup_{t \in G(g)(u)} G(f)(t) \\ &= (G(f) \circ G(g))(u) , \end{aligned}$$

and therefore the equality $G(g \circ f) = G(f) \circ G(g)$.

The functors are mutually inverse

It remains to check that the functors F and G are mutually inverse.

We have for every finite set S the sequences of equalities

$$G(F(S)) = G(S) = S , \quad F(G(S)) = F(S) = S ,$$

which tells us that F and G are mutually inverse on objects.

Let θ be a morphism in Γ from a set S to a set T. We have for all $s \in S, t \in T$ the sequence of equivalences

$$
\begin{aligned}
& s \in (GF)(\theta)(t) \\
\iff & s \in G(F(\theta))(t) \\
\iff & s \in F(\theta)^{-1}(t) \\
\iff & F(\theta)(s) = t \\
\iff & s \in \theta(t)\,.
\end{aligned}
$$

This shows that $(GF)(\theta) = \theta$ for every morphism θ in Γ.

Let $f : S \to T$ be a partially defined function between sets S and T. We have for all $s \in S, t \in T$ the sequence of equivalences

$$
\begin{aligned}
& (FG)(f)(s) = t \\
\iff & F(G(f))(s) = t \\
\iff & s \in G(f)(t) \\
\iff & s \in f^{-1}(t) \\
\iff & f(s) = t\,.
\end{aligned}
$$

This shows that $(FG)(f) = f$ for every morphism f in Fin^∂.

We have thus shown that the two functors F and G are mutually inverse.

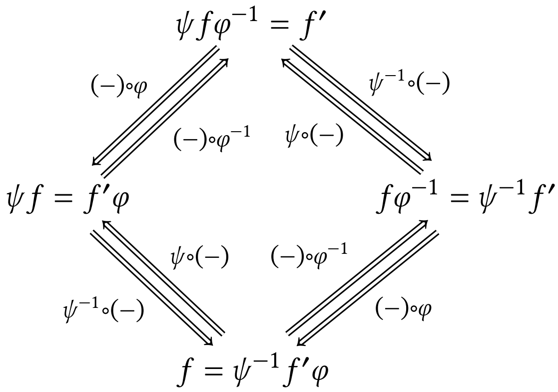
Exercise 1.5.iii

We denote the given isomorphisms by $\varphi : a \to a'$ and $\psi : b \to b'$.

The commutativities of the four square diagrams are equivalent to the following four equations respectively:

$$
\psi f \varphi^{-1} = f'\,, \quad \psi f = f' \varphi\,, \quad f \varphi^{-1} = \psi^{-1} f'\,, \quad f = \psi^{-1} f' \varphi\,.
$$

These four equations are equivalent because ψ and φ are isomorphisms:



I hence suffices to show for one of these four equations that there exists a unique morphism $f' : a' \to b'$ satisfying this equation. We can use the first equation, as already explained in the textbook.

Exercise 1.5.iv

(i)

Let $f : x \to y$ be a morphism in C for which the morphism $Ff : Fx \to Fy$ is an isomorphism in D. This means that there exists a morphism $g : Fy \to Fx$ in D with both $Ff \cdot g = 1_{Fy}$ and $g \cdot Ff = 1_{Fx}$.

There exists a morphism $f' : y \to x$ in C with $g = Ff'$ because the functor F is full. It follows from the sequence of equalities

$$F(f \cdot f') = Ff \cdot Ff' = Ff \cdot g = 1_{Fy} = F1_y$$

that $f \cdot f' = 1_y$ because the functor F is faithful. We find in the same way that also $f' \cdot f = 1_x$. This shows that f is an isomorphism with inverse f'.

We have thus proven that the inverse of Ff in D lifts uniquely to an inverse of f in C. This entails that f is an isomorphism.

(ii)

That the two objects Fx and Fy in D are isomorphic means that there exists an isomorphism $g : Fx \to Fy$ in D. There exists a morphism $f : x \to y$ in C with $g = Ff$ because the functor F is full. It follows from part (i) of this exercise that f is again an isomorphism. The existence of this isomorphism shows that the two objects x and y are isomorphic.

Exercise 1.5.v

Let P be the partially ordered set consisting of the two natural numbers 0 and 1 with the usual ordering $0 < 1$. Let Q be the preordered set consisting of two distinct elements x and y with both $x \leq y$ and $y \leq x$. Let P and Q be the categories corresponding to P and Q respectively. (The category P is $\mathbb{1}$, and the category Q is $\mathbb{I}$.)

The map

$$P \longrightarrow Q\,, \quad 0 \longmapsto x\,, \quad 1 \longmapsto y$$

is isotone, and therefore describes a functor $F : \mathsf{P} \to \mathsf{Q}$. The functor F is faithful because there exist no two distinct parallel morphisms in P. (More generally, every functor from a preordered set into any other category is faithful.)

The unique arrow $0 \to 1$ in P is not an isomorphism. But its image under F, which is the unique arrow $x \to y$ in Q, is an isomorphism. The functor F therefore doesn't reflect isomorphisms.

Exercise 1.5.vi

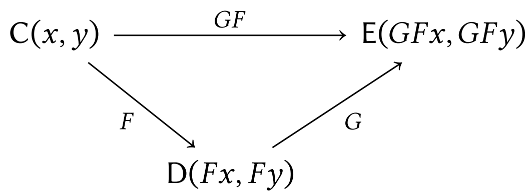
(i)

We show the following result:

Proposition 1.F. Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$ be two functors.

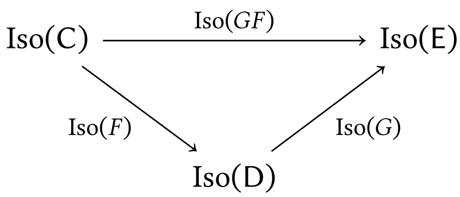
1. If both F and G are faithful, then their composite GF is again faithful.
2. If both F and G are full, then their composite GF is again full.
3. If both F and G are essentially surjective, then their composite GF is again essentially surjective.

Proof. We observe for every two objects x and y in C the following commutative diagram of sets and functions between these sets:



1. Let x and y be any two objects of C. The maps $F\colon \mathsf{C}(x,y) \to \mathsf{D}(Fx,Fy)$ and $G\colon \mathsf{D}(Fx,Fy) \to \mathsf{E}(GFx,GFy)$ are both injective, whence their composite $GF\colon \mathsf{C}(x,y) \to \mathsf{E}(GFx,GFy)$ is again injective.
2. Let x and y be any two objects of C. The maps $F\colon \mathsf{C}(x,y) \to \mathsf{D}(Fx,Fy)$ and $G\colon \mathsf{D}(Fx,Fy) \to \mathsf{E}(GFx,GFy)$ are both surjective, whence their composite $GF\colon \mathsf{C}(x,y) \to \mathsf{E}(GFx,GFy)$ is again surjective.
3. For every category X we denote by $\operatorname{Iso}(\mathsf{X})$ the collection of isomorphisms classes of objects of X. Functors preserve isomorphisms, whence every functor $H\colon \mathsf{X} \to \mathsf{Y}$ induces a function $\operatorname{Iso}(H)\colon \operatorname{Iso}(\mathsf{X}) \to \operatorname{Iso}(\mathsf{Y})$ given by $\operatorname{Iso}(H)([x]) = [Hx]$.

   In the given situation we have the following commutative diagram:



   The functions $\operatorname{Iso}(F)$ and $\operatorname{Iso}(G)$ are both surjective by assumption. Their composite $\operatorname{Iso}(G) \circ \operatorname{Iso}(F) = \operatorname{Iso}(GF)$ is therefore again surjective. ∎

(ii)

According to Theorem 1.5.9 there exist functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{E}$ both of which are full, faithful, and essentially surjective. It follows from Proposition 1.F that their composite $GF\colon \mathsf{C} \to \mathsf{E}$ is again full, faithful, and essentially surjective. Once again according to Theorem 1.5.9, this means that C and E are equivalent.

We have thus proven that equivalence of categories is transitive.

Every category is equivalent to itself via its identity functor. Equivalence of categories is therefore reflexive.

Two categories C and D are equivalent if and only if there exist functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ with $GF \cong 1_{\mathsf{C}}$ and $FG \cong 1_{\mathsf{D}}$. This formulation is symmetric in C and D, whence equivalence of categories is symmetric.

Exercise 1.5.vii

Let more specifically x be an object of G and let G be the automorphism group of x, i.e., the group $\mathrm{Aut}_{\mathsf{C}}(x)$. Let F be the inclusion functor from BG to G, given on objects by $F{*} = x$ and on morphisms by $Fg = g$ for every $g \in G$.

There exist for every object y of G an isomorphism $\varphi_y : x \to y$ in G because G is a connected groupoid. We choose φ_x as 1_x.

We define a functor F' from G to BG as follows. On objects, we set $F'y \coloneqq *$ for every object y of G (the only possible choice). We define for every two objects y and z in G the action of F' on the set $\mathsf{G}(y,z)$ as the map

$$\mathsf{G}(y,z) \xrightarrow{\varphi_z^{-1}\circ(-)\circ\varphi_y} \mathsf{G}(x,x) = G = \mathrm{B}G(*,*) = \mathrm{B}G(F'y, F'z)\,.$$

In other words,

$$F'f \coloneqq \varphi_z^{-1} f \varphi_y$$

for every morphism $f : y \to z$ in G. We need to check that the assignment F' is indeed functorial:

- We have for every object y of G the sequence of equalities

  $$F'1_y = \varphi_y^{-1} 1_y \varphi_y = \varphi_y^{-1}\varphi_y = 1_x = 1_* = 1_{F'y}\,.$$

  This shows that F' preserves identity morphisms.

- We have for any two morphisms $f_1 : y \to z$ and $f_2 : z \to w$ in G the sequence of equalities

  $$F'f_2 \cdot F'f_1 = \varphi_w^{-1} f_2 \varphi_z \cdot \varphi_z^{-1} f_1 \varphi_y = \varphi_w^{-1} f_2 f_1 \varphi_y = F'(f_2 f_1)\,.$$

  This shows that F' preserves composition of morphisms.

We have thus shown that the assignment F' is functorial.

The composite $F'F$ is the identity functor on $\mathrm{B}G$: we have for the single object $*$ of $\mathrm{B}G$ the sequence of equalities

$$F'F* = F'x = * \,,$$

and we have for every morphism $g \colon * \to *$ in $\mathrm{B}G$ the sequence of equalities

$$F'Fg = F'g = \varphi_x^{-1} g \varphi_x = 1_x^{-1} g 1_x = 1_x g 1_x = g \,.$$

The composite FF' won't be the identity functor on G (unless x is the only object in G, in which case F is an isomorphism with inverse F'), but it will be isomorphic to this identity functor. We claim more specifically that the family $\varphi \coloneqq (\varphi_y)_y$, where y ranges over the objects of G, is a natural isomorphism from FF' to 1_{G}.

We already know that φ is an isomorphism in each component, and that φ_y goes from $x = FF'y$ to $y = 1_{\mathrm{C}} y$ for every object y of G. It therefore only remains to check the naturality of φ. To this end, we need to check that for every morphism $f \colon y \to z$ in G the following square diagram commutes:

$$\begin{array}{ccc} FF'y & \xrightarrow{FF'f} & FF'z \\ {\scriptstyle \varphi_y}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_z} \\ y & \xrightarrow{\quad f \quad} & z \end{array}$$

This diagram commutes because

$$\varphi_z \cdot FF'f = \varphi_z \cdot F(\varphi_z^{-1} f \varphi_y) = \varphi_z \cdot (\varphi_z^{-1} f \varphi_y) = \varphi_z \varphi_z^{-1} f \varphi_y = f \varphi_y \,.$$

We have altogether shown that F' is an essential inverse to F.

Exercise 1.5.viii

The author doesn't know enough geometry for this exercise.

Exercise 1.5.ix

Let C be a locally small category and let D be a category equivalent to C. This means that there exists an equivalence of categories F from D to C. The functor F is both full and faithful by Theorem 1.5.9, and therefore induces for every two objects d and d' of D the bijection

$$\mathsf{D}(d, d') \xrightarrow{F} \mathsf{C}(Fd, Fd') .$$

We know that $\mathsf{C}(Fd, Fd')$ is a set because the category C is locally small. Consequently, $\mathsf{D}(d, d')$ is also a set.

Exercise 1.5.x

Let D be a discrete category and let C be a category that is equivalent to D. More explicitly, let $F : \mathsf{D} \to \mathsf{C}$ be an equivalence of categories.

The category D is in particular a groupoid. The category C is therefore also a groupoid, since by Exercise 1.5.iv the equivalence F reflect isomorphisms.

We have for every two objects d and d' of D the induced bijection

$$\mathsf{D}(d, d') \xrightarrow{F} \mathsf{C}(Fd, Fd') .$$

We know that the set $\mathsf{C}(c, c')$ contains at most one element for every two objects c and c' of C. Consequently, $\mathsf{D}(d, d')$ consists of at most one element. This entails that for every object d of D, the automorphism group of d is trivial.

From these observations and from Exercise 1.5.vii, we altogether find that D is the disjoint union of its connected components, each of which is equivalent to $\mathrm{B}1$.[3] (In terms of graphs one might picture each connected component as a complete graph. An edge in this graph represents a mutually inverse pair of non-identity isomorphisms.)

Exercise 1.5.xi

The forgetful functor $\mathsf{Ab} \to \mathsf{Group}$

Let F be the inclusion functor from Ab to Group.

The functor F is full and faithful because Ab is a full subcategory of Group. The functor F is not essentially surjective because no non-abelian group is isomorphic to an abelian group. (And non-abelian groups do, in fact, exist.)

We see in particular that F is not an equivalence of categories.

[3] Where 1 denotes the trivial group.

The forgetful functor Ring → Ab

Let F be the forgetful functor from Ring to Ab.

The functor F is faithful. But it is not full because not every additive map between rings is a homomorphism of rings, as not every additive map between rings is multiplicative.

The functor F is not essentially surjective because not every abelian group can be endowed with the structure of a unitary ring. Consider, for example, the abelian group $A := \mathbb{Q}/\mathbb{Z}$. A ring structure on A would consist of a multiplication map

$$A \otimes_{\mathbb{Z}} A \longrightarrow A\,, \quad a \otimes b \longmapsto ab\,.$$

But the group $A \otimes_{\mathbb{Z}} A$ is trivial. The only $\mathbb{Z}$-bilinear multiplication on A is therefore the zero multiplication, which is non-unital.

We find in particular that the functor F is not an equivalence of categories.

The functor $(-)^\times \colon$ Ring → Group

Let F be the functor $(-)^\times \colon$ Ring → Group that assigns to each ring its group of invertible elements (also known as its group of units).

The functor F is not faithful. To see this, let R be an integral domain, let a and b be two distinct elements of R, and let $\varphi, \psi \colon R[x] \to R$ be the two evaluation homomorphisms determined by $\varphi(x) = a$ and $\psi(x) = b$. The group $R[x]^\times$ is simply $R^\times$ because R is an integral domain, and both φ and ψ induce the identity homomorphism on $R^\times$.

The functor F is also not full: while there exists no homomorphism of rings from $\mathbb{F}_3$ to $\mathbb{F}_5$, there nevertheless exists a homomorphism of groups from $\mathbb{F}_3^\times \cong \mathbb{Z}/2$ to $\mathbb{F}_5^\times \cong \mathbb{Z}/4$; even a non-trivial one.

Regarding the essential surjectivity of F, we need to examine if every group can occur as the group of units of some ring. This is not the case, as explained in [MSE19].

The functor F is in particular not an equivalence of categories.

The forgetful functor Ring → Rng

Let F be the forgetful functor from Ring to Rng.

The functor F is faithful because Ring is a subcategory of Rng. The functor F is not full because there exists a homomorphism from the zero ring to $\mathbb{Z}$ in Rng, but not in Ring.

The functor F is also not essentially surjective, because no non-unitary ring is isomorphic to a unitary ring.

The forgetful functor Field → Ring

Let F be the forgetful functor from Field to Ring.

The functor F is full and faithful because Field is a full subcategory of Ring. The functor F is not essentially surjective because every non-field ring, e.g., $\mathbb{Z}$, is not isomorphic to a field.

The forgetful functor $\mathsf{Mod}_R \to \mathsf{Ab}$

Let F be the forgetful functor from Mod_R to Ab.

The functor F is faithful for every ring R. Whether the functor F is full depends on the ring R:

- The functor F will typically not be full, because an additive map between R-modules is not necessarily R-linear.
- If R is either a quotient or localization of $\mathbb{Z}$, then every additive map between R-modules is already R-linear.
- Suppose more generally that the unique homomorphism of rings from $\mathbb{Z}$ to R is an epimorphism.

  It then follows that the two canonical homomorphisms of rings from R to $R \otimes_{\mathbb{Z}} R$ are equal, since they are equal after pre-composition with the homomorphism $\mathbb{Z} \to R$. In other words, $r \otimes 1 = 1 \otimes r$ in $R \otimes_{\mathbb{Z}} R$ for every $r \in R$.

  Let now M and N be two R-modules and let $f : M \to N$ be an additive map. For $m \in M$ we can then consider the auxiliary map

  $$h : R \otimes_{\mathbb{Z}} R \longrightarrow N\,, \quad r_1 \otimes r_2 \longmapsto r_1 f(r_2 m)$$

  because f is additive. We find that

  $$r f(m) = h(r \otimes 1) = h(1 \otimes r) = f(rm)$$

  for every $r \in R$. This shows that the map f is already R-linear.

  Our argumentation is essentially taken from [MSE18a]. A ring for which the unique homomorphism of rings $\mathbb{Z} \to R$ is an epimorphism is called **solid**. More information and references about solid rings can be found at [MSE21].

The functor F is essentially surjective if and only if each abelian group can be endowed with an R-module structure. This is the case if and only if the unique homomorphism rings from $\mathbb{Z}$ to R splits, i.e., if and only if there exists a homomorphism of rings from R to $\mathbb{Z}$:

- Suppose that such a homomorphism of rings $\varphi \colon R \to \mathbb{Z}$ exists. For every abelian group A we can then pull back the unique $\mathbb{Z}$-module structure on A along φ to an R-module structure on A.
- Suppose conversely that every abelian group A can be endowed with the structure of an R-module. This means that we have for every abelian group A a homomorphism of rings from R to $\mathrm{End}_{\mathbb{Z}}(A)$. For $A = \mathbb{Z}$ we have $\mathrm{End}_{\mathbb{Z}}(A) = \mathbb{Z}$, and therefore a homomorphism of rings from R to $\mathbb{Z}$.

Moreover, there exists a homomorphism of rings from R to $\mathbb{Z}$ if and only if the ring R is of the form $R \cong R' \rtimes \mathbb{Z}$ for some possibly non-unitary ring R'. That is, if and only if R is the unitalization of R'.

The functor F typically won't be an equivalence.

1.6 The art of the diagram chase

Exercise 1.6.i

Let C be a category, let i be an initial object in C, let t be a terminal object in C, and let f be a morphism from t to i.

There exists a unique morphism g from i to t because i is initial in C (and also because t is terminal in C). The composite fg is a morphism from i to i, and there exists only one such morphism in C because i is initial. The identity morphism 1_i is also a morphism from i to i, so by uniqueness we must have $fg = 1_i$. By using that t is terminal in C, we can similarly see that $gf = 1_t$.

This shows that f is an isomorphism with inverse g.

Exercise 1.6.ii

Let C be a category and let t and t' be two terminal objects in C. There exists a unique morphism f from t to t' in C because t' is terminal in C. We can see in at least two ways that f is an isomorphism:

First argumentation

There exists a morphism g from t' to t in C because t is terminal in C. The composite gf is a morphism from t to t, and 1_t is also a morphism from t to t. But we know that there exists only one morphism from t to t because t is terminal in C. We hence find that $gf = 1_t$. We can see in the same way that also $fg = 1_{t'}$. This then tells us that f is an isomorphism with inverse g.

Second argumentation

We observe that the induced map $f_* : \mathrm{C}(c,t) \to \mathrm{C}(c,t')$ is bijective for every object c of C because the sets $\mathrm{C}(c,t)$ and $\mathrm{C}(c,t')$ are both singletons. By Lemma 1.2,3, the morphism f is an isomorphism.

Exercise 1.6.iii

Let $F : \mathrm{C} \to \mathrm{D}$ be a faithful functor. Let $f : x \to y$ be a morphism in C for which the induced morphism $Ff : Fx \to Fy$ in D is a monomorphism.

Faithful functors reflect monomorphisms, first solution

Let $g_1, g_2 : w \to x$ be two morphisms in C with $fg_1 = fg_2$. Then also

$$Ff \cdot Fg_1 = F(fg_1) = F(fg_2) = Ff \cdot Fg_2 \,,$$

and thus $Fg_1 = Fg_2$ because Ff is a monomorphism. It then further follows that $g_1 = g_2$ because the functor F is faithful.

Faithful functors reflect monomorphisms, second solution

We have for every object c of C the following commutative square diagram:

$$\begin{array}{ccc} \mathrm{C}(c,x) & \xrightarrow{\;f_*\;} & \mathrm{C}(c,y) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathrm{D}(Fc,Fx) & \xrightarrow{\;(Ff)_*\;} & \mathrm{D}(Fc,Fy) \end{array}$$

In this diagram, both vertical arrows are injective because the functor F is faithful, and the lower horizontal arrow is injective because Ff is a monomorphism. It follows from the commutativity of the diagram that the composite

$$\mathsf{C}(c, x) \xrightarrow{f_*} \mathsf{C}(c, y) \xrightarrow{F} \mathsf{D}(Fc, Fy)$$

is injective, which entails that the map $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is injective. As this holds for every object c of C, this shows that f is a monomorphism.

Faithful functors reflect epimorphisms

The covariant functor $F : \mathsf{C} \to \mathsf{D}$ is also a covariant functor from C^{op} to D^{op}, and still faithful. We thus have for every morphism f in C the logical steps

$$\begin{aligned} & Ff \text{ is an epimorphism in } \mathsf{D} \\ \iff & Ff \text{ is a monomorphism in } \mathsf{D}^{\mathrm{op}} \\ \implies & f \text{ is a monomorphism in } \mathsf{C}^{\mathrm{op}} \\ \iff & f \text{ is an epimorphism in } \mathsf{C}\,. \end{aligned}$$

Monomorphisms and epimorphisms in concrete categories

Let C be a concrete category with forgetful functor $U : \mathsf{C} \to \mathsf{Set}$. We have for every morphism f in C the logical steps

$$\begin{aligned} & Uf \text{ is injective} \\ \iff & Uf \text{ is a monomorphism in } \mathsf{Set} \\ \implies & f \text{ is a monomorphism in } \mathsf{C}\,, \end{aligned}$$

as well as the logical steps

$$\begin{aligned} & Uf \text{ is surjective} \\ \iff & Uf \text{ is an epimorphism in } \mathsf{Set} \\ \implies & f \text{ is an epimorphism in } \mathsf{C}\,, \end{aligned}$$

because the functor U is faithful.

Exercise 1.6.iv

Let $f : x \to y$ be a morphism in a category C that is not a monomorphism, and whose domain x and codomain y are distinct. Let C′ be the subcategory of C consisting of the two objects x and y, their identity morphisms 1_x and 1_y, and the morphism f. The inclusion functor U from C′ to C is faithful, because C′ is a subcategory of C, and the morphism f is a monomorphism in C′. But Uf is not a monomorphism in C by choice of f.

To see that faithful functors need not preserve epimorphisms we can consider the inclusion functor from $(\mathrm{C}')^{\mathrm{op}}$ to C^{op}.

Exercise 1.6.v

We can adapt our argumentation from the previous exercise.

A non-injective monomorphism

We can consider the non-injective map

$$f : \{1, 2\} \longrightarrow * ,$$

and the subcategory of Set whose objects are the two sets $\{*\}$ and $\{1, 2\}$ and whose morphisms are $1_{\{*\}}$, $1_{\{1,2\}}$ and f.

A non-surjective epimorphism

We can similarly consider the non-surjective function

$$g : \{*\} \longrightarrow \{1, 2\}, \quad * \longmapsto 1 ,$$

and the subcategory of Set whose objects are the two sets $\{*\}$ and $\{1, 2\}$ and whose morphisms are $1_{\{*\}}$, $1_{\{1,2\}}$ and g.

Exercise 1.6.vi

We denote the category of (C, T)-coalgebras by $\mathrm{CoAlg}(\mathrm{C}, T)$. Every morphism in $\mathrm{CoAlg}(\mathrm{C}, T)$ is also a morphism in C. The composition of morphisms in $\mathrm{CoAlg}(\mathrm{C}, T)$ is the composition of morphisms in C. For every coalgebra (c, γ), the identity morphism of (c, γ) in $\mathrm{CoAlg}(\mathrm{C}, T)$ is the identity morphism of c in C, i.e., $1_{(c,\gamma)} = 1_c$.

We first observe that every coalgebra (c, γ) gives rise to another coalgebra, namely $(Tc, T\gamma)$. Moreover, γ is a morphism of coalgebras from (c, γ) to $(Tc, T\gamma)$ by the commutativity of the following diagram:

$$\begin{array}{ccc} c & \xrightarrow{\ \gamma\ } & Tc \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle T\gamma} \\ Tc & \xrightarrow{\ T\gamma\ } & T^2c \end{array}$$

Suppose now that (c, γ) is a terminal coalgebra, i.e., a terminal object in the category $\mathsf{CoAlg}(\mathsf{C}, T)$. We know that γ is a morphism of coalgebras from (c, γ) to $(Tc, T\gamma)$. We also know that there exists a (unique) morphism of coalgebras

$$\delta : (Tc, T\gamma) \longrightarrow (c, \gamma)$$

because the coalgebra (c, γ) is terminal. This means that δ is a morphism from Tc to c in C, so that the following diagram commutes:

$$\begin{array}{ccc} Tc & \xrightarrow{\ \delta\ } & c \\ {\scriptstyle T\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ T^2c & \xrightarrow{\ T\delta\ } & Tc \end{array} \tag{1.4}$$

The composite $\delta\gamma$ is a morphism of coalgebras from (c, γ) to itself. As (c, γ) is terminal, this morphism must be the identity morphism of (c, γ). Therefore, $\delta\gamma = 1_{(c,\gamma)} = 1_c$. It further follows from the commutativity of the diagram (1.4) that

$$\gamma\delta = T\delta \cdot T\gamma = T(\delta\gamma) = T1_c = 1_{Tc} = 1_{(Tc,T\gamma)} \,.$$

This shows that γ and δ are mutually inverse isomorphisms of coalgebras.

1.7 The 2-category of categories

Exercise 1.7.i

Let C be a small category and let D be a locally small category. Let F and G be two functors from C to D, and let $\mathrm{Nat}(F, G)$ be the collection of natural transformations from F to G. The map

$$\mathrm{Nat}(F,G) \longrightarrow \prod_{c \in \mathsf{C}} \mathsf{D}(Fc, Gc)\,, \quad \alpha \longmapsto (\alpha_c)_c$$

is injective, and its codomain is a small product of sets, and therefore again a set. It follows that the collection $\mathrm{Nat}(F,G)$ is also a set.

Exercise 1.7.ii

Let $f : x \to y$ be a morphism in C. The morphism f in C induces the morphism

$$Ff : Fx \longrightarrow Fy$$

in D. It follows from the naturality of β that the resulting square diagram

$$\begin{array}{ccc} HFx & \xrightarrow{HFf} & HFy \\ {\scriptstyle \beta_{Fx}}\downarrow & & \downarrow{\scriptstyle \beta_{Fy}} \\ KFx & \xrightarrow{KFf} & KFy \end{array}$$

in E commutes. By applying the functor L to this diagram, we get the commutative square diagram

$$\begin{array}{ccc} LHFx & \xrightarrow{LHFf} & LHFy \\ {\scriptstyle L\beta_{Fx}}\downarrow & & \downarrow{\scriptstyle L\beta_{Fy}} \\ LKFx & \xrightarrow{LKFf} & LKFy \end{array}$$

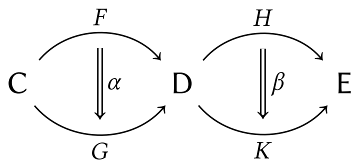
in F. We have $L\beta_{Fx} = (L\beta F)_x$ and similarly $L\beta_{Fy} = (L\beta F)_y$, and can therefore rewrite this commutative square diagram as follows:

$$\begin{array}{ccc} LHFx & \xrightarrow{LHFf} & LHFy \\ {\scriptstyle (L\beta F)_x}\downarrow & & \downarrow{\scriptstyle (L\beta F)_y} \\ LKFx & \xrightarrow{LKFf} & LKFy \end{array}$$

That this diagram commutes for every morphism $f : x \to y$ in C means precisely that the family $L\beta F = (L\beta_c F)_c$ is a natural transformation from LHF to LKF.

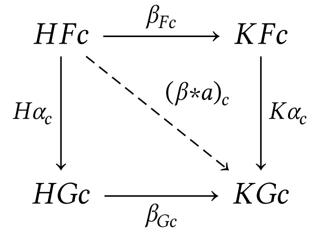
Exercise 1.7.iii

Consider the following diagram of functors and natural transformations:



(1.5)

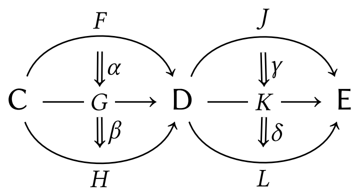
The vertical composite of α and β was originally defined componentwise as the diagonal morphism in the following commutative diagram:



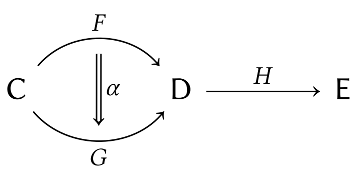
We also know that the horizontal and vertical composition satisfy the interchange rule

$$(\delta \cdot \gamma) * (\beta \cdot \alpha) = (\delta * \beta) \cdot (\gamma * \alpha)$$

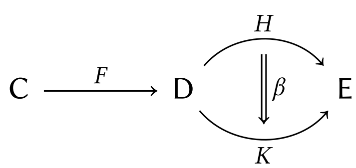
whenever we are in the following situation:



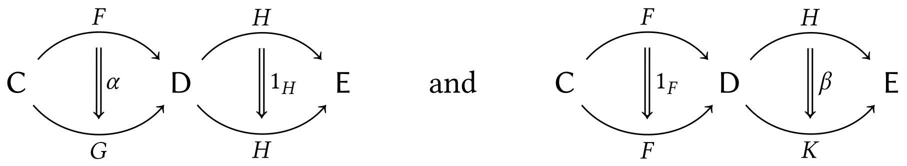
We have also seen that from the horizontal composition of natural transformation we can derive the whiskering of natural transformations: in the situations



and



we have the induced diagrams



(1.6)

and therefore the natural transformations

$$H\alpha \coloneqq 1_H * \alpha \qquad \text{and} \qquad \beta F \coloneqq \beta * 1_F \,.$$

However, we can also conversely express horizontal composition via vertical composition and whiskering, as required by this exercise: in the situation of the diagram (1.5) we have the sequence of equalities

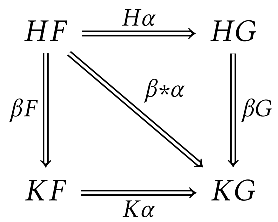
$$\beta * \alpha = (1_K \cdot \beta) * (\alpha \cdot 1_F) = (1_K * \alpha) \cdot (\beta * 1_F) = K\alpha \cdot \beta F \,,$$

as well as the sequence of equalities

$$\beta * \alpha = (\beta \cdot 1_H) * (1_G \cdot \alpha) = (\beta * 1_G) \cdot (1_H * \alpha) = \beta G \cdot H\alpha \,.$$

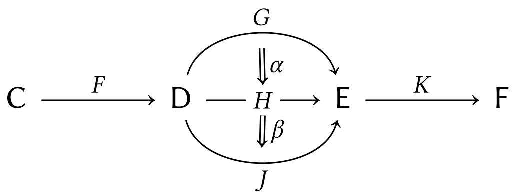
These equations give us two ways of expressing horizontal composition via vertical composition and whiskering.

We can express the overall situation by the following commutative diagram of natural transformations:



Exercise 1.7.iv

Lemma 1.G (Whiskering and vertical composition I). In the situation



we have the identity

$$K(\beta \cdot \alpha)F = K\beta F \cdot K\alpha F\,.$$

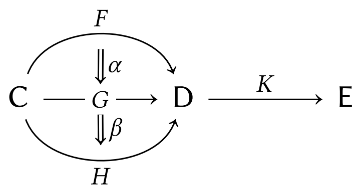
Proof. We have for every object c of C the sequence of equalities

$$\begin{aligned}(K(\beta \cdot \alpha)F)_c &= K(\beta \cdot \alpha)_{Fc} \\ &= K(\beta_{Fc} \cdot \alpha_{Fc}) \\ &= K\beta_{Fc} \cdot K\alpha_{Fc} \\ &= (K\beta F)_c \cdot (K\alpha F)_c \\ &= (K\beta F \cdot K\alpha F)_c\,,\end{aligned}$$

and therefore altogether the equality $K(\beta \cdot \alpha)F = K\beta F \cdot K\alpha F$. ∎

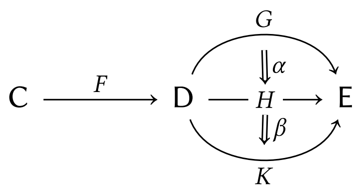
Corollary 1.H (Whiskering and vertical composition II). Let C, D and E be three categories.

1. In the situation



   we have the equality $K(\beta \cdot \alpha) = K\beta \cdot K\alpha$.

2. In the situation

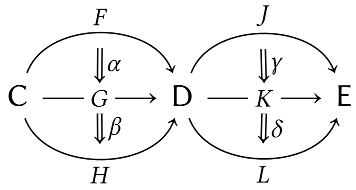


   we have the equality $(\beta \cdot \alpha)F = \beta F \cdot \alpha F$. ∎

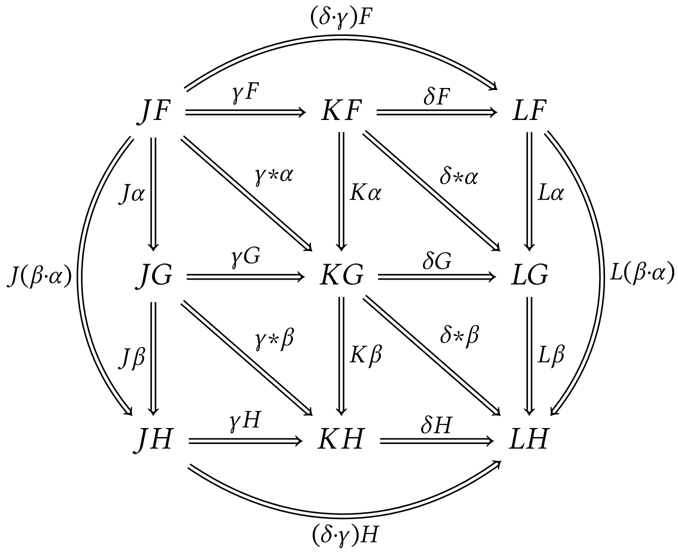
Proof.

1. We have $K(\beta \cdot \alpha) = K(\beta \cdot \alpha)1_{\mathsf{C}} = K\beta 1_{\mathsf{C}} \cdot K\alpha 1_{\mathsf{C}} = K\beta \cdot K\alpha$.
2. We have $(\beta \cdot \alpha)F = 1_{\mathsf{E}}(\beta \cdot \alpha)F = 1_{\mathsf{E}}\beta F \cdot 1_{\mathsf{E}}\alpha F = \beta F \cdot \alpha F$. ∎

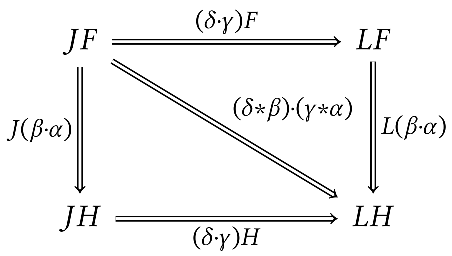
We now return to the given situation:



The diagram



commutes by definition of the horizontal composition of natural transformations (the four inner squares) and by Corollary 1.H (the four outer parts). The above diagram has the following subdiagram:



But the composite of the upper horizontal arrow and right-side vertical arrow is given by

$$L(\beta \cdot \alpha) \cdot (\delta \cdot \gamma)F = (\delta \cdot \gamma) * (\beta \cdot \alpha)\,.$$

(The same goes for the composite of the left-side vertical arrow with the lower horizontal arrow.) Consequently,

$$(\delta * \beta) \cdot (\gamma * \alpha) = (\delta \cdot \gamma) * (\beta \cdot \alpha) .$$

Exercise 1.7.v

Let D be a category and let d be an object of D. Let $\mathrm{End}(d)$ denote the collection of endomorphisms of d in D. For every two such endomorphisms f and g, their composite gf is again an endomorphism of D. The composition of morphisms in D therefore restricts to a binary operation on $\mathrm{End}(d)$, making $\mathrm{End}(d)$ into a magma. As composition of morphism in D is associative, the restricted operation on $\mathrm{End}(d)$ is again associative, upgrading $\mathrm{End}(d)$ to a semigroup. The identity morphism 1_d acts as a neutral element in $\mathrm{End}(d)$, further upgrading $\mathrm{End}(d)$ to a monoid.

Let now C be a category and let D be the functor category C^{C}. It follows from the above argumentation that $\mathrm{End}_{\mathsf{D}}(1_{\mathsf{C}})$ is a monoid under vertical composition of natural transformations. It follows from the upcoming proposition that $\mathrm{End}_{\mathsf{D}}(1_{\mathsf{C}})$ is also a monoid under horizontal composition of natural transformation. These two monoid structures on $\mathrm{End}_{\mathsf{D}}(1_{\mathsf{C}})$ satisfy the interchange property from Lemma 1.7.7. It follows from the Eckmann–Hilton argument that both monoid structures agree and are commutative.

Lemma 1.I (Whiskering with identity functors). Let $F, G : \mathsf{C} \to \mathsf{D}$ be two parallel functors and let $\alpha : F \Rightarrow G$ be a natural transformation. Then $\alpha 1_{\mathsf{C}} = \alpha$ and $1_{\mathsf{D}} \alpha = \alpha$.

Proof. For the first equality we observe that $\alpha 1_{\mathsf{C}}$ is a natural transformation from $F 1_{\mathsf{C}}$ to $G 1_{\mathsf{C}}$, and thus again a natural transformation from F to G. We also have

$$(\alpha 1_{\mathsf{C}})_c = \alpha_{(1_{\mathsf{C}} c)} = \alpha_c$$

for every object c of C, and thus overall $\alpha 1_{\mathsf{C}} = \alpha$.

For the second equality we observe that $1_{\mathsf{D}} \alpha$ is a natural transformation from $1_{\mathsf{D}} F$ to $1_{\mathsf{D}} G$, and thus again a natural transformation from F to G. We also have

$$(1_{\mathsf{D}} \alpha)_c = 1_{\mathsf{D}} \alpha_c = \alpha_c$$

for every object c of C, and thus overall $1_{\mathsf{D}} \alpha = \alpha$. ∎

Lemma 1.J (Whiskering of the identity natural transformation). Let

$$F\colon \mathsf{C} \longrightarrow \mathsf{D} \quad \text{and} \quad G\colon \mathsf{D} \longrightarrow \mathsf{C}$$

be two functors. Then $G1_F = 1_{GF}$ and $1_G F = 1_{GF}$.

Proof. We have for every object c of C the sequences of equalities

$$(G1_F)_c = G(1_F)_c = G1_{Fc} = 1_{GFc} = (1_{GF})_c$$

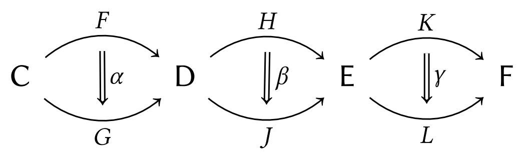
and

$$(1_G F)_c = (1_G)_{Fc} = 1_{GFc} = (1_{GF})_c \,,$$

and thus altogether $G1_F = 1_{GF}$ and $1_G F = 1_{GF}$. ∎

Proposition 1.K (Assocativity and units for horizontal composition).

1. In the situation



   we have the equality $(\gamma * \beta) * \alpha = \gamma * (\beta * \alpha)$ of natural transformations from KHF to LJG.

2. We have for every natural transformation $\alpha\colon F \Rightarrow G$ between two functors $F, G\colon \mathsf{C} \to \mathsf{D}$ the equalities

$$\alpha * 1_{1_\mathsf{C}} = \alpha \quad \text{and} \quad 1_{1_\mathsf{D}} * \alpha = \alpha \,.$$[4]

Proof. We prove both claims independently of one another.

1. The horizontal composition $\gamma * \beta$ is a natural transformation from KH to LJ, and the horizontal composition $\beta * \alpha$ is a natural transformation from HF

[4] Here 1_{1_C} denotes the identity natural transformation of the identity functor of C. Note that $\alpha * 1_{1_\mathsf{C}}$ is a natural transformation from $F1_\mathsf{C} = F$ to $G1_\mathsf{C} = G$, and that similarly $1_{1_\mathsf{D}} * \alpha$ is a natural transformation from $1_\mathsf{D} F = F$ to $1_\mathsf{D} G = G$.

to JG. We have therefore the sequences of equalities

$$\begin{aligned}(\gamma * \beta) * \alpha &= (\gamma * \beta)\operatorname{codom}(\alpha) \cdot \operatorname{dom}(\gamma * \beta)\alpha \\ &= (\gamma * \beta)\operatorname{codom}(\alpha) \cdot \operatorname{dom}(\gamma)\operatorname{dom}(\beta)\alpha \\ &= (\gamma * \beta)G \cdot KH\alpha \\ &= (\gamma \operatorname{codom}(\beta) \cdot \operatorname{dom}(\gamma)\beta)G \cdot KH\alpha \\ &= (\gamma J \cdot K\beta)G \cdot KH\alpha \\ &= \gamma JG \cdot K\beta G \cdot KH\alpha\end{aligned}$$

and similarly

$$\begin{aligned}\gamma * (\beta * \alpha) &= \gamma \operatorname{codom}(\beta * \alpha) \cdot \operatorname{dom}(\gamma)(\beta * \alpha) \\ &= \gamma \operatorname{codom}(\beta)\operatorname{codom}(\alpha) \cdot \operatorname{dom}(\gamma)(\beta * \alpha) \\ &= \gamma JG \cdot K(\beta * \alpha) \\ &= \gamma JG \cdot K(\beta \operatorname{codom}(\alpha) \cdot \operatorname{dom}(\beta)\alpha) \\ &= \gamma JG \cdot K(\beta G \cdot H\alpha) \\ &= \gamma JG \cdot K\beta G \cdot KH\alpha\,,\end{aligned}$$

by Corollary 1.H.[5]

2. It follows from Lemma 1.I and Lemma 1.J that

$$\begin{aligned}\alpha * 1_{1_{\mathsf{C}}} &= \alpha \operatorname{codom}(1_{1_{\mathsf{C}}}) \cdot \operatorname{dom}(\alpha)1_{1_{\mathsf{C}}} \\ &= \alpha 1_{\mathsf{C}} \cdot F1_{1_{\mathsf{C}}} \\ &= \alpha \cdot 1_{F1_{\mathsf{C}}} \\ &= \alpha \cdot 1_F \\ &= \alpha\end{aligned}$$

[5]More generally, given functors $F_i, G_i : \mathsf{C}_i \to \mathsf{C}_{i+1}$ and natural transformations $\alpha_i : F_i \to G_i$ for $i = 1, \dots, n$, we have the identity

$$\alpha_n * \cdots * \alpha_1 = (\alpha_n G_{n-1} G_{n-2} \cdots G_1) \cdot \cdots \cdot (F_n \cdots F_3 \alpha_2 G_1) \cdot (F_n \cdots F_3 F_2 \alpha_1)\,.$$

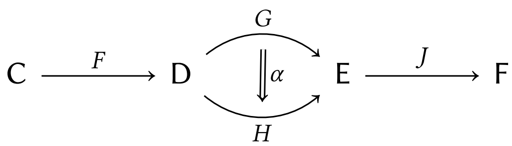
and

$$\begin{aligned} 1_{1_{\mathsf{D}}} * \alpha &= 1_{1_{\mathsf{D}}} \operatorname{codom}(\alpha) \cdot \operatorname{dom}(1_{1_{\mathsf{D}}})\alpha \\ &= 1_{1_{\mathsf{D}}} G \cdot 1_{\mathsf{D}} \alpha \\ &= 1_{1_{\mathsf{D}} G} \cdot \alpha \\ &= 1_G \cdot \alpha \\ &= \alpha \,. \end{aligned}$$

This shows the claimed equalities. ∎

Exercise 1.7.vi

Lemma 1.L. Suppose that in the situation



the natural transformation α is an isomorphism. Then the whiskered natural transformation $J\alpha F$ is again an isomorphism.

Proof. The component $(J\alpha F)_c = J\alpha_{Fc}$ is an isomorphism for every object c of C because each component of α is an isomorphism and because the functor J preserves isomorphisms. ∎

The composites

$$1_{\mathsf{C}} \xrightarrow{\eta} GF = G1_{\mathsf{D}}F \xrightarrow{G\eta' F} GG'F'F$$

and

$$F'FGG' \xrightarrow{F'\varepsilon G'} F'1_{\mathsf{D}}G' = F'G' \xrightarrow{\varepsilon'} 1_{\mathsf{E}}$$

are again natural isomorphisms thanks to Lemma 1.L.

Exercise 1.7.vii

From a bifunctor $\mathsf{C} \times \mathsf{D} \to \mathsf{E}$ **to functor** $\mathsf{C} \to \mathsf{E}^{\mathsf{D}}$

Let $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ be a bifunctor.

We have for every object c of C the associated inclusion functor

$$I_c : \mathsf{D} \longrightarrow \mathsf{C} \times \mathsf{D}, \quad d \longmapsto (c, d), \quad g \longmapsto (1_c, g).$$

The composite FI_c is again a functor. This composite is precisely $F(c, -)$, whence we have found that $F(c, -)$ is indeed a functor. It is given on objects by

$$F(c, -)(d) = F(c, d),$$

and on morphisms by

$$F(c, -)(g) = F(1_c, g).$$

To describe how $F(c, -)$ depends on C, let $f : c \to c'$ be a morphism in C. We have for every morphism $g : d \to d'$ in D the following commutative square diagram in $\mathsf{C} \times \mathsf{D}$:

$$\begin{array}{ccc}
(c, d) & \xrightarrow{(1_c, g)} & (c, d') \\
{\scriptstyle (f, 1_d)} \downarrow & \overset{(f, g)}{\searrow} & \downarrow {\scriptstyle (f, 1_{d'})} \\
(c', d) & \xrightarrow[(1_{c'}, g)]{} & (c', d')
\end{array}$$

By applying the functor F to this commutative square diagram, we arrive at the following commutative square diagram in E:

$$\begin{array}{ccc}
F(c, d) & \xrightarrow{F(1_c, g)} & F(c, d') \\
{\scriptstyle F(f, 1_d)} \downarrow & & \downarrow {\scriptstyle F(f, 1_{d'})} \\
F(c', d) & \xrightarrow[F(1_{c'}, g)]{} & F(c', d')
\end{array}$$

Denoting $F(f, 1_d)$ as $F(f, -)_d$, we can rewrite this commutative diagram as follows:

$$\begin{array}{ccc} F(c,-)(d) & \xrightarrow{F(c,-)(g)} & F(c,-)(d') \\ {\scriptstyle F(f,-)_d}\downarrow & & \downarrow{\scriptstyle F(f,-)_{d'}} \\ F(c',-)(d) & \xrightarrow[F(c',-)(g)]{} & F(c',-)(d') \end{array}$$

The commutativity of this square diagram tells us that the family

$$F(f,-) := (F(f,-)_d)_{d \in \mathsf{D}}$$

is a natural transformation from the functor $F(c,-)$ to the functor $F(c',-)$.

It remains to show that $F(f,-)$ is functorial in f. More precisely, we need to show that

$$F(1_c,-) = 1_{F(c,-)}$$

for every object c of C, and that

$$F(f'f,-) = F(f',-) \cdot F(f,-)$$

for all composable morphisms $f : c \to c'$ and $f' : c' \to c''$ in C. The first equality holds true because

$$F(1_c,-)_d = F(1_c,1_d) = F1_{(c,d)} = 1_{F(c,d)} = 1_{F(c,-)(d)} = (1_{F(c,-)})_d$$

for every object d of D. The second equality holds true because

$$\begin{aligned} F(f'f,-)_d &= F(f'f,1_d) \\ &= F(f'f,1_d 1_d) \\ &= F((f',1_d) \cdot (f,1_d)) \\ &= F(f',1_d) \cdot F(f,1_d) \\ &= F(f',-)_d \cdot F(f,-)_d \\ &= (F(f',-) \cdot F(f,-))_d \end{aligned}$$

for every object d of D.

We have overall shown that a bifunctor $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ results in a functor $F' : \mathsf{C} \to \mathsf{E}^{\mathsf{D}}$ given on objects by $F'(c) = F(c,-)$ and on morphisms by $F'(f) = F(f,-)$.

From a functor $\mathsf{C} \to \mathsf{E}^{\mathsf{D}}$ to a bifunctor $\mathsf{C} \times \mathsf{D} \to \mathsf{E}$

Let now G be a functor from C to E^{D}.

For every object (c, d) of $\mathsf{C} \times \mathsf{D}$ let

$$G'(c, d) := G(c)(d) .$$

We have for every morphism $(f, g) : (c, d) \to (c', d')$ in $\mathsf{C} \times \mathsf{D}$ the induced natural transformation $G(f) : G(c) \Rightarrow G(c')$, and hence the following commutative square diagram:

$$\begin{CD} G(c)(d) @>{G(c)(g)}>> G(c)(d') \\ @V{G(f)_d}VV @VV{G(f)_{d'}}V \\ G(c')(d) @>{G(c')(g)}>> G(c')(d') \end{CD}$$

We let $G'(f, g)$ be the diagonal morphism in this diagram, which is a morphism in E from $G'(c, d)$ to $G'(c', d')$.

We claim that the assignment G' is a functor from $\mathsf{C} \times \mathsf{D}$ to E. It remains to verify the functoriality of G'.

- We have to show that $G'(1_{(c,d)}) = 1_{G'(c,d)}$ for every object (c, d) in $\mathsf{C} \times \mathsf{D}$. We have $1_{(c,d)} = (1_c, 1_d)$, whence the morphism $G'(1_{(c,d)})$ is defined as the diagonal morphism in the following diagram:

$$\begin{CD} G(c)(d) @>{G(c)(1_d)}>> G(c)(d) \\ @V{G(1_c)_d}VV @VV{G(1_c)_d}V \\ G(c)(d) @>{G(c)(1_d)}>> G(c)(d) \end{CD}$$

  The functor value $G(c)$ is itself a functor, from D to E, whence

$$G(c)(1_d) = 1_{G(c)(d)} = 1_{G'(c,d)} .$$

  Similarly, the natural transformation $G(1_c)$ equals $1_{G(c)}$ by the functoriality of G, whence

$$G(1_c)_d = (1_{G(c)})_d = 1_{G(c)(d)} = 1_{G'(c,d)} .$$

We can altogether rewrite the above square diagram as follows:

$$\begin{array}{ccc} G'(c,d) & \xrightarrow{1_{G'(c,d)}} & G'(c,d) \\ {\scriptstyle 1_{G'(c,d)}}\big\downarrow & & \big\downarrow{\scriptstyle 1_{G'(c,d)}} \\ G'(c,d) & \xrightarrow{1_{G'(c,d)}} & G'(c,d) \end{array}$$

We see that the diagonal morphism in this diagram is $1_{G'(c,d)}$.

- We also need to show that $G'((f',g')\cdot(f,g)) = G'(f',g')\cdot G'(f,g)$ for every two composable morphisms

$$(f,g)\colon (c,d) \longrightarrow (c',d') \quad \text{and} \quad (f',g')\colon (c',d') \longrightarrow (c'',d'')$$

in $\mathsf{C}\times\mathsf{D}$. We consider the following commutative diagram:

$$\begin{array}{ccccc} G(c)(d) & \xrightarrow{G(c)(g)} & G(c)(d') & \xrightarrow{G(c)(g')} & G(c)(d'') \\ {\scriptstyle G(f)_d}\big\downarrow & \overset{G'(f,g)}{\searrow} & \big\downarrow{\scriptstyle G(f)_{d'}} & \overset{G'(f,g')}{\searrow} & \big\downarrow{\scriptstyle G(f)_{d''}} \\ G(c')(d) & \xrightarrow{G(c')(g)} & G(c')(d') & \xrightarrow{G(c')(g')} & G(c')(d'') \\ {\scriptstyle G(f')_d}\big\downarrow & \overset{G'(f',g)}{\searrow} & \big\downarrow{\scriptstyle G(f')_{d'}} & \overset{G'(f',g')}{\searrow} & \big\downarrow{\scriptstyle G(f')_{d''}} \\ G(c'')(d) & \xrightarrow{G(c'')(g)} & G(c'')(d') & \xrightarrow{G(c'')(g')} & G(c'')(d'') \end{array}$$

Leaving out the middle node of this diagram, we get the following commutative subdiagram:

$$\begin{array}{ccc} G(c)(d) & \xrightarrow{G(c)(g')\cdot G(c)(g)} & G(c)(d'') \\ {\scriptstyle G(f')_d\cdot G(f)_d}\big\downarrow & \overset{G'(f',g')\cdot G(f,g)}{\searrow} & \big\downarrow{\scriptstyle G(f')_{d''}\cdot G(f)_{d''}} \\ G(c'')(d) & \xrightarrow[G(c'')(g')\cdot G(c'')(g)]{} & G(c'')(d'') \end{array}$$

It follows from the functoriality of $G(c)$ that the upper horizontal arrow can be simplified as $G(c)(g'g)$. Similarly, the lower horizontal arrow can be simplified as $G(c'')(g'g)$. The vertical arrow on the left-hand side can be rewritten as $(G(f') \cdot G(f))_d$, and thus as $G(f'f)_d$ by the functoriality of G. Similarly, the vertical arrow on the right-hand side can be rewritten as $G(f'f)_{d''}$. We get overall the following commutative diagram:

$$\begin{array}{ccc} G(c)(d) & \xrightarrow{G(c)(g'g)} & G(c)(d'') \\ {\scriptstyle G(f'f)_d}\downarrow & \searrow^{G'(f',g')G(f,g)} & \downarrow{\scriptstyle G(f'f)_{d''}} \\ G(c'')(d) & \xrightarrow[G(c'')(g'g)]{} & G(c'')(d'') \end{array}$$

The morphism $G'((f', g') \cdot (f, g)) = G'(f'f, g'g)$ is defined as the diagonal morphism in precisely this commutative square diagram. Consequently, this morphism agrees with the composite $G'(f', g') \cdot G'(f, g)$.

The constructions are mutually inverse

It remains to show that the two constructions are mutually inverse.

First part Let first F be a bifunctor from $\mathsf{C} \times \mathsf{D}$ to E, let F' be the induced functor from C to E^{D}, and let F'' be the induced bifunctor from $\mathsf{C} \times \mathsf{D}$ to E.

We have for every object (c, d) of $\mathsf{C} \times \mathsf{D}$ the sequence of equalities

$$F''(c, d) = F'(c)(d) = F(c, -)(d) = F(c, d) .$$

For every morphism $(f, g) \colon (c, d) \to (c', d')$ in $\mathsf{C} \times \mathsf{D}$, the morphism $F''(f, g)$ is defined as the diagonal morphism in the following commutative square diagram:

$$\begin{array}{ccc} F'(c)(d) & \xrightarrow{F'(c)(g)} & F'(c)(d') \\ {\scriptstyle F'(f)_d}\downarrow & & \downarrow{\scriptstyle F'(f)_{d'}} \\ F'(c')(d) & \xrightarrow{F'(c')(g)} & F'(c')(d') \end{array}$$

Using the definition of F', this diagram can be rewritten as follows:

$$\begin{array}{ccc} F(c,d) & \xrightarrow{F(1_c,g)} & F(c,d') \\ {\scriptstyle F(f,1_d)}\big\downarrow & & \big\downarrow{\scriptstyle F(f,1_{d'})} \\ F(c',d) & \xrightarrow{F(1_{c'},g)} & F(c',d') \end{array}$$

The diagonal morphism in this diagram is given by

$$F(f,1_{d'}) \cdot F(1_c,g) = F((f,1_{d'}) \cdot (1_c,g)) = F(f1_c, 1_{d'}g) = F(f,g)\,.$$

This shows altogether that $F''(f,g)$ equals $F(f,g)$.

Second part Let now G be a functor from C to E^{D}. Let G' be the induced bifunctor from $\mathsf{C} \times \mathsf{D}$ to E, and let G'' be the induced functor from C to D^{E}. We want to show that $G'' = G$. To this end we need to show that the two functors G'' and G agree both on objects and on morphisms.

We first show that $G''(c) = G(c)$ for every object c of C, i.e., that the two functors G and G'' agree on objects. We need to show that the two functors $G''(c)$ and $G(c)$ from D to E agree both on objects and on morphisms.

- Let d be an arbitrary object of D. We have the sequence of equalities

  $$G''(c)(d) = G'(c,d) = G(c)(d)\,.$$

  This tells us that $G''(c)$ and $G(c)$ agree on objects.
- Let $g : d \to d'$ be a morphism in D. The morphism $G''(c)(g)$ is defined as $G'(1_c, g)$, which in turn is defined as the diagonal morphism in the following commutative diagram:

  $$\begin{array}{ccc} G(c)(d) & \xrightarrow{G(c)(g)} & G(c)(d') \\ {\scriptstyle G(1_c)_d}\big\downarrow & & \big\downarrow{\scriptstyle G(1_c)_{d'}} \\ G(c)(d) & \xrightarrow{G(c)(g)} & G(c)(d') \end{array}$$

  We know from the functoriality of G that

  $$G(1_c)_d = (1_{G(c)})_d = 1_{G(c)(d)}$$

It follows that the diagonal morphism in the above diagram is

$$G(c)(g) \cdot G(1_c)_d = G(c)(g) \cdot 1_{G(c)(d)} = G(c)(g)\,.$$

This shows that $G''(c)(g) = G(c)(g)$, so that $G''(c)$ and $G(c)$ agree on morphisms.

This shows altogether that the two functors G'' and G agree on objects.

We now show that the two functors G'' and G agree on morphisms. To this end, let $f : c \to c'$ be a morphism in C. We need to show that the two natural transformations $G''(f)$ and $G(f)$ from $G''(c) = G(c)$ to $G''(c') = G(c')$ are equal. We hence need to show that $G''(f)_d = G(f)_d$ for every object d of D. The morphism $G''(f)_d$ is defined as $G'(f, 1_d)$, which in turn is defined as the diagonal morphism in the following commutative diagram:

$$\begin{array}{ccc} G(c)(d) & \xrightarrow{G(c)(1_d)} & G(c)(d) \\ {\scriptstyle G(f)_d}\downarrow & & \downarrow{\scriptstyle G(f)_d} \\ G(c')(d) & \xrightarrow{G(c')(1_d)} & G(c')(d) \end{array}$$

We know from the functoriality of $G(c)$ that $G(c)(1_d) = 1_{G(c)(d)}$. The diagonal morphism in the above diagram is therefore given by

$$G(f)_d \cdot G(c)(1_d) = G(f)_d \cdot 1_{G(c)(d)} = G(f)_d\,,$$

as desired.

Chapter 2

Universal Properties, Representability, and the Yoneda Lemma

2.1 Representable functors

Exercise 2.1.i

We denote the given functors by

$$I_0 : \mathbb{1} \longrightarrow \mathbb{2}\,, \quad 0 \longmapsto 0\,,$$
$$I_1 : \mathbb{1} \longrightarrow \mathbb{2}\,, \quad 0 \longmapsto 1\,,$$

as well as

$$P : \mathbb{2} \longrightarrow \mathbb{1}\,, \quad 0, 1 \longmapsto 0\,, \quad i \longmapsto 1_0\,,$$

where $i : 0 \to 1$ denotes the unique non-identity morphism in $\mathbb{2}$. These functors induce morphisms in Cat, i.e., natural transformations,

$$I_0^* : \mathrm{Cat}(\mathbb{2}, -) \longrightarrow \mathrm{Cat}(\mathbb{1}, -)\,,$$
$$I_1^* : \mathrm{Cat}(\mathbb{2}, -) \longrightarrow \mathrm{Cat}(\mathbb{1}, -)\,,$$
$$P^* : \mathrm{Cat}(\mathbb{1}, -) \longrightarrow \mathrm{Cat}(\mathbb{2}, -)\,.$$

Under the canonical isomorphisms $\mathrm{Cat}(\mathbb{2}, -) \cong \mathrm{mor}$ and $\mathrm{Cat}(\mathbb{1}, -) \cong \mathrm{ob}$ these natural transformation correspond to natural transformations

$$\alpha, \beta : \mathrm{mor} \longrightarrow \mathrm{ob}\,, \quad \gamma : \mathrm{ob} \longrightarrow \mathrm{mor}\,.$$

The natural transformation α

Let $f\colon x \to y$ be a morphism in a small category C. Under the isomorphism $\mathrm{Cat}(\mathbb{2}, \mathrm{C}) \cong \operatorname{mor} \mathrm{C}$ the morphism f corresponds to the functor

$$F\colon \mathbb{2} \longrightarrow \mathrm{C}\,, \quad 0 \longmapsto x\,, \quad 1 \longmapsto y\,, \quad i \longmapsto f\,.$$

The resulting functor $I_0^*(F) = FI_0$ is given by

$$FI_0\colon \mathbb{1} \longrightarrow \mathrm{C}\,, \quad 0 \longmapsto x\,.$$

Under the isomorphism $\mathrm{Cat}(\mathbb{1}, \mathrm{C}) \cong \operatorname{ob} \mathrm{C}$ the functor FI_0 corresponds to the object x.

We find overall that the component $\alpha_{\mathrm{C}}\colon \operatorname{mor} \mathrm{C} \to \operatorname{ob} \mathrm{C}$ assigns to each morphism its domain.

The natural transformation β

We find in the same way that for every small category C the component β_{C} assigns to each morphism in C its codomain.

The natural transformation γ

Let x be an object of a small category C. The corresponding functor under the isomorphism $\mathrm{Cat}(\mathbb{1}, \mathrm{C}) \cong \operatorname{ob} \mathrm{C}$ is given by

$$F\colon \mathbb{1} \longrightarrow \mathrm{C}\,, \quad 0 \longmapsto x\,.$$

The resulting functor $P^*(F) = PF$ is given by

$$PF\colon \mathbb{2} \longrightarrow \mathrm{C}\,, \quad 0, 1 \longmapsto x\,, \quad i \longmapsto 1_x\,.$$

Under the isomorphism $\mathrm{Cat}(\mathbb{2}, \mathrm{C}) \cong \operatorname{mor} \mathrm{C}$ the functor PF corresponds to the morphism 1_x.

We find overall that the component $\gamma_{\mathrm{C}}\colon \operatorname{ob} \mathrm{C} \to \operatorname{mor} \mathrm{C}$ assigns to each object its identity morphism.

Exercise 2.1.ii

Representable functors preserve monomorphisms

There exists by assumption an object c of C for which the two functors F and $\mathrm{C}(c,-)$ are isomorphic. Let α be a natural isomorphism from $\mathrm{C}(c,-)$ to F.

Let $f : x \to y$ be a monomorphism in C. This entails that the induced map $f_* : \mathrm{C}(c,x) \to \mathrm{C}(c,y)$ is injective. We thus have the commutative square diagram

$$\begin{array}{ccc} \mathrm{C}(c,x) & \xrightarrow{f_*} & \mathrm{C}(c,y) \\ {\scriptstyle\alpha_x}\downarrow & & \downarrow{\scriptstyle\alpha_y} \\ F(x) & \xrightarrow{F(f)} & F(y) \end{array}$$

in which both vertical arrows are bijections and the upper horizontal arrow is injective. It follows that

$$F(f) = \alpha_y f_* \alpha_x^{-1}$$

is a composite of injections, and therefore itself injective. This shows that the functor F preserves monomorphisms, as the monomorphisms in Set are precisely those maps that are injective.

A non-representable functor

For every group G let $C(G)$ be its set of conjugacy classes. Every homomorphism of groups $\varphi : G \to H$ induces a map $C\varphi : C(G) \to C(H)$ that assigns to the conjugacy class of an element g the conjugacy class of the image element $\varphi(g)$. The assignment C is a functor from Group to Set.

Let S_3 be the symmetric group on three letters. This group contains two elements ρ_1 and ρ_2 of order 3, which are conjugated to one another. The two elements ρ_1 and ρ_2 are related via $\rho_2 = \rho_1^2$.

Let i be the inclusion homomorphism from $\mathbb{Z}/3$ to S_3 given by $i([1]) = \rho_1$. Then also

$$i([2]) = i(2 \cdot [1]) = i([1])^2 = \rho_1^2 = \rho_2 \,.$$

The group $\mathbb{Z}/3$ is abelian, whence each element of $\mathbb{Z}/3$ forms its own conjugacy class. The homomorphism i therefore maps the two non-conjugated elements [1] and [2] of $\mathbb{Z}/3$ onto the two conjugated elements ρ_1 and ρ_2 of S_3. This tells us that the induced map Ci is not injective.

We find that the functor C does not preserve monomorphisms. It therefore cannot be representable.

Exercise 2.1.iii

We first observe that parts (i) and (ii) are equivalent:

Let K be an essential inverse to H. We have the sequence of isomorphisms

$$FK \cong GHK \cong G1_{\mathsf{D}} \cong G \,.$$

We hence find that parts (i) and (ii) are interchangeable via

$$\mathsf{C} \longleftrightarrow \mathsf{D}\,, \quad F \longleftrightarrow G\,, \quad H \longleftrightarrow K\,.$$

In the following, we prove part (i).

That G is representable tells us that there exists an object d of D such that $G \cong \mathsf{D}(d, -)$. There exists an object c of C with $d \cong H(c)$ because the functor H is an equivalence, and thus essentially surjective. It follows that

$$F \cong GH \cong \mathsf{D}(d,-)H \cong \mathsf{D}(d, H(-)) \cong \mathsf{D}(H(c), H(-)) \cong \mathsf{C}(c, -)\,.$$

This shows that the functor F is represented by the object c.

Exercise 2.1.iv

A subfunctor F of $\mathsf{C}(c,-)$ consists of a subset $F(x)$ of $\mathsf{C}(c,x)$ for every object x of C such that

$$g_*(F(x)) \subseteq F(y)$$

for every morphism $g \colon x \to y$ in C. In other words, we have a collection of distinguished morphisms that is closed under post-composition with arbitrary other morphisms.

We have not been able to find a better, or more explicit description of such subfunctors.

Exercise 2.1.v

The category $\mathbb{I}$ consists of two objects, named 0 and 1, a morphism i from 0 to 1 and a morphism j from 1 to 0 (and the two identity morphisms). We have for every category C a bijection given by

$$\mathrm{Cat}(\mathbb{I}, \mathsf{C}) \longrightarrow \{\text{isomorphisms in } \mathsf{C}\}, \quad F \longmapsto Fi\,.$$

The category $\mathbb{1}$ is the subcategory of $\mathbb{I}$ that consists of both objects and the morphism i (and the two identity morphisms). We have for every category C a bijection given by

$$\mathrm{Cat}(\mathbb{1}, \mathsf{C}) \longrightarrow \{\text{morphisms in } \mathsf{C}\}, \quad F \longmapsto Fi\,.$$

We see that for every category C the inclusion

$$\{\text{isomorphisms in } \mathsf{C}\} \subseteq \{\text{morphisms in } \mathsf{C}\}$$

corresponds to the pullback map

$$I^* : \mathrm{Cat}(\mathbb{I}, \mathsf{C}) \longrightarrow \mathrm{Cat}(\mathbb{1}, \mathsf{C})$$

induced by the inclusion functor $I : \mathbb{1} \to \mathbb{I}$.

2.2 The Yoneda lemma

Exercise 2.2.i

The dual version of the Yoneda lemma is as follows:

> For every contravariant functor $F : \mathsf{C} \to \mathsf{Set}$, whose domain C is locally small, and every object c of C, the map
>
> $$\Phi : \mathrm{Hom}(\mathsf{C}(-, c), F) \longrightarrow Fc\,, \quad \alpha \longmapsto \alpha_c(1_c)$$
>
> is bijective and natural in both c and F.

Injectivity of Φ

To see that the map Φ is injective let $\alpha : \mathsf{C}(-,c) \Rightarrow F$ be a natural isomorphism, let x be an arbitrary object of C and let f be an arbitrary element of $\mathsf{C}(-,c)(x)$. As $\mathsf{C}(-,c)(x) = \mathsf{C}(x,c)$, this means that f is a morphism from x to c in C. It follows from the commutativity of the square diagram

$$\begin{array}{ccc} \mathsf{C}(c,c) & \xrightarrow{f^*} & \mathsf{C}(x,c) \\ {\scriptstyle\alpha_c}\downarrow & & \downarrow{\scriptstyle\alpha_x} \\ Fc & \xrightarrow{Ff} & Fx \end{array}$$

that

$$\alpha_x(f) = \alpha_x(1_c \cdot f) = \alpha_x(f^*(1_c)) = (Ff)(\alpha_c(1_c)) = (Ff)(\Phi(\alpha)) .$$

This shows that the entire natural transformation α is uniquely determined by the single element $\Phi(\alpha)$.

Surjectivity of Φ

Let u be an arbitrary element of the set Fc. We consider for every object x of C the map

$$\alpha_x : \mathsf{C}(x,c) \longrightarrow Fx , \quad g \longmapsto (Fg)(u) .$$

(Note that if g is a morphism from x to c in C, then Fg is a map from Fc to Fx, whence $(Fg)(u)$ is a well-defined element of the set Fx.) We claim that the family $\alpha := (\alpha_x)_{x \in \mathsf{C}}$ is a natural transformation from $\mathsf{C}(-,c)$ to F with $\Phi(\alpha) = u$.

To see that α is a natural transformation from $\mathsf{C}(-,c)$ to F, we consider an arbitrary morphism $f : x \to y$ in C and the resulting diagram:

$$\begin{array}{ccc} \mathsf{C}(y,c) & \xrightarrow{f^*} & \mathsf{C}(x,c) \\ {\scriptstyle\alpha_y}\downarrow & & \downarrow{\scriptstyle\alpha_x} \\ Fy & \xrightarrow{Ff} & Fx \end{array}$$

This diagram commutes because

$$\alpha_x(f^*(h)) = \alpha_x(hf) = F(hf)(u) = (Ff)((Fh)(u)) = (Ff)(\alpha_y(h))$$

for every element h of $\mathsf{C}(y, c)$, i.e., every morphism $h\colon c \to y$ in C. This shows the naturality of α.

We also have $\Phi(\alpha) = \alpha_c(1_c) = (F1_c)(u) = 1_{F(c)}(u) = u$, as desired.

Naturality in c

We relabel the bijection Φ as Φ_c, and want to show that Φ_c is natural in c.

Let $f : c \to d$ be a morphism in C. We need to show that the square diagram

$$\begin{array}{ccc} \operatorname{Hom}(\mathsf{C}(-,d),F) & \xrightarrow{(f_*)^*} & \operatorname{Hom}(\mathsf{C}(-,c),F) \\ {\scriptstyle\Phi_d}\big\downarrow & & \big\downarrow{\scriptstyle\Phi_c} \\ Fd & \xrightarrow{\quad Ff \quad} & Fc \end{array} \tag{2.1}$$

commutes. To this end let α be an element in the top-left corner of this diagram, i.e., a natural transformation from $\mathsf{C}(-, d)$ to F. One path from the top-left corner to the bottom-right corner in the diagram (2.1) equals

$$\begin{aligned} \Phi_c((f_*)^*(\alpha)) &= \Phi_c(\alpha \cdot f_*) \\ &= (\alpha \cdot f_*)_c(1_c) \\ &= (\alpha_c \cdot (f_*)_c)(1_c) \\ &= \alpha_c((f_*)_c(1_c)) \\ &= \alpha_c(f \cdot 1_c) \\ &= \alpha_c(f)\,. \end{aligned}$$

For the other path from the top-left corner to the bottom-right corner we observe that the diagram

$$\begin{array}{ccc} \mathsf{C}(d,d) & \xrightarrow{f^*} & \mathsf{C}(c,d) \\ {\scriptstyle\alpha_d}\big\downarrow & & \big\downarrow{\scriptstyle\alpha_c} \\ Fd & \xrightarrow{Ff} & Fc \end{array}$$

commutes by the naturality of α, whence

$$(Ff)(\Phi_d(\alpha)) = (Ff)(\alpha_d(1_d)) = \alpha_c(f^*(1_d)) = \alpha_c(1_d \cdot f) = \alpha_c(f)\,.$$

This shows the commutativity of the diagram (2.1).

Naturality in F

We relabel the bijection Φ as Φ_F, and want to show that Φ_F is natural in F.

Let F and G be two contravariant functors from C to Set, and let $\beta : F \Rightarrow G$ be a natural transformation. We need to show that the square diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathsf{C}(-,c), F) & \xrightarrow{\beta_*} & \mathrm{Hom}(\mathsf{C}(-,c), G) \\ \Big\downarrow{\scriptstyle \Phi_F} & & \Big\downarrow{\scriptstyle \Phi_G} \\ Fc & \xrightarrow{\beta_c} & Gc \end{array}$$

commutes. To this end let α be an element of the top-left corner of this diagram, i.e., let α be a natural transformation from $\mathsf{C}(-,c)$ to F. The sequence of equalities

$$\Phi_G(\beta_*(\alpha)) = \Phi_G(\beta \cdot \alpha) = (\beta \cdot \alpha)_c(1_c) = (\beta_c \cdot \alpha_c)(1_c) = \beta_c(\alpha_c(1_c)) = \beta_c(\Phi_F(\alpha))$$

tells us that the diagram indeed commutes.

Exercise 2.2.ii

Why should it?

Exercise 2.2.iii

The functor category $\mathsf{C} := \mathsf{Set}^{(\omega^{\mathrm{op}})}$ can be regarded as the category of contravariant functors from ω to Set. This category can (up to isomorphism) more explicitly be described as follows:

- The objects of C are diagrams of the form

$$A_0 \xleftarrow{a_0} A_1 \xleftarrow{a_1} A_2 \xleftarrow{a_2} A_3 \longleftarrow \cdots$$

  consisting of sets and maps between them.
- A morphism from an object $((A_n)_n, (a_n)_n)$ to an object $((B_n)_n, (b_n)_n)$ is a sequence $(f_n)_n$ of maps $f_n : A_n \to B_n$ such that the following diagram com-

mutes:

$$\begin{array}{ccccccccc}
A_0 & \xleftarrow{a_0} & A_1 & \xleftarrow{a_1} & A_2 & \xleftarrow{a_2} & A_3 & \longleftarrow & \cdots \\
{\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \\
B_0 & \xleftarrow{b_0} & B_1 & \xleftarrow{b_1} & B_2 & \xleftarrow{b_2} & B_3 & \longleftarrow & \cdots
\end{array}$$

For every object n of ω, the object $\text{よ}(n) = \omega(-, n)$ consists of $n + 1$ many singleton sets, followed by empty sets:[1]

$$\{*\} \longleftarrow \{*\} \longleftarrow \cdots \longleftarrow \{*\} \longleftarrow \varnothing \longleftarrow \varnothing \longleftarrow \cdots$$

We know that in ω there exists a unique morphism from n to m if $n \leq m$, and no morphism if $n > m$. It suffices to show that the same holds true for the induced diagrams $\text{よ}(n)$ and $\text{よ}(m)$.

- Suppose that $n \leq m$. Then there exists precisely one sequence of maps $(f_n)_n$ that makes the following diagram commute:

$$\begin{array}{lcccccccccccccc}
\text{よ}(n): & \{*\} & \longleftarrow \cdots \longleftarrow & \{*\} & \longleftarrow & \varnothing & \longleftarrow \cdots \longleftarrow & \varnothing & \longleftarrow & \varnothing & \longleftarrow \cdots \\
& {\scriptstyle f_0}\downarrow & \iddots & {\scriptstyle f_n}\downarrow & & {\scriptstyle f_{n+1}}\downarrow & \iddots & {\scriptstyle f_m}\downarrow & & {\scriptstyle f_{m+1}}\downarrow & \\
\text{よ}(m): & \{*\} & \longleftarrow \cdots \longleftarrow & \{*\} & \longleftarrow & \{*\} & \longleftarrow \cdots \longleftarrow & \{*\} & \longleftarrow & \varnothing & \longleftarrow \cdots
\end{array}$$

  Indeed, the maps $f_0, \dots, f_n$ are necessarily the identity maps on the singleton set, and f_k for $k > n$ is necessarily the empty map.

- Suppose now that $n > m$. A morphism $f : \text{よ}(n) \to \text{よ}(m)$ contains a map $f_{m+1} : \{*\} \to \varnothing$, but such a map does not exist. Consequently, no morphism from $\text{よ}(n)$ to $\text{よ}(m)$ exists.

Exercise 2.2.iv

We know that a natural transformation is a natural isomorphism if and only if it is an isomorphism in each component. The isomorphisms in Set are precisely the bijective maps. A natural transformation between Set-valued

[1] We denote the Yoneda embedding by よ instead of y.

functors is therefore a natural isomorphism if and only if it is bijective in each component.

We also know that the induced transformations f_* and f^* in parts (ii) and (iii) are always natural.

This exercise is therefore just a reformulation of Lemma 1.2.3.

Exercise 2.2.v

The set Ω consists of two elements, whence it admits $2^2 = 4$ maps into itself. These maps are as follows:

1. The identity map.
2. The transposition that swaps $\top$ and $\bot$.
3. The constant map with value $\top$.
4. The constant map with value $\bot$.

We denote these maps by 1_Ω, σ, $c_\top$ and $c_\bot$ respectively.

The natural isomorphism between $P(X)$ and $\mathrm{Set}(X, \Omega)$ is for every set X given as follows:

- For every function $\chi \colon X \to \Omega$, the corresponding subset of X is the preimage $\chi^{-1}(\top)$.
- For every subset A of X, the corresponding function is its characteristic function

$$\chi_A \colon X \longrightarrow \Omega, \quad x \longmapsto \begin{cases} \top & \text{if } x \in A, \\ \bot & \text{if } x \notin A. \end{cases}$$

We denote this natural isomorphism from $\mathrm{Set}(-, \Omega)$ to P by α.

1. The natural endomorphism $(1_\Omega)_*$ of $\mathrm{Set}(-, \Omega)$ is $1_{\mathrm{Set}(-,\Omega)}$. The corresponding endomorphism of P is given by

$$\alpha \cdot (1_\Omega)_* \cdot \alpha^{-1} = \alpha 1_{\mathrm{Set}(-,\Omega)} \alpha^{-1} = \alpha\alpha^{-1} = 1_P \, .$$

   In other words, the identity map induces the identity endomorphism.

2. The natural endomorphism of P induced by σ is given by $\alpha\sigma_*\alpha^{-1}$, and its components are given by

$$\begin{aligned}(\alpha \cdot \sigma_* \cdot \alpha^{-1})_X(A) &= \alpha_X((\sigma_*)_X(\alpha_X^{-1}(A))) \\ &= \alpha_X((\sigma_*)_X(\chi_A)) \\ &= \alpha_X(\sigma\chi_A) \\ &= (\sigma\chi_A)^{-1}(\top) \\ &= \chi_A^{-1}(\sigma^{-1}(\top)) \\ &= \chi_A^{-1}(\bot) \\ &= X \setminus A\,.\end{aligned}$$

   In other words, it is given by taking complements.

3. The natural endomorphism of P induced by $c_\top$ is given by $\alpha(c_\top)_*\alpha^{-1}$, and its components are given by

$$\begin{aligned}(\alpha \cdot (c_\top)_* \cdot \alpha^{-1})_X(A) &= \cdots \\ &= \chi_A^{-1}(c_\top^{-1}(\top)) \\ &= \chi_A^{-1}(\Omega) \\ &= X\,.\end{aligned}$$

   In other words, it is given by mapping each subset to the entire set.

4. The natural endomorphism of P induced by $c_\bot$ is given by $\alpha(c_\bot)_*\alpha^{-1}$, and its components are given by

$$\begin{aligned}(\alpha \cdot (c_\bot)_* \cdot \alpha^{-1})_X(A) &= \cdots \\ &= \chi_A^{-1}(c_\bot^{-1}(\top)) \\ &= \chi_A^{-1}(\emptyset) \\ &= \emptyset\,.\end{aligned}$$

   In other words, it is given by mapping each subset to the empty set.

Exercise 2.2.vi

Remark 2.A. The formulation of the question is slightly misleading: an "endomorphism of the category of spaces" is a functor Top → Top, and there are many such functors (e.g., constant functors). But the question then asks for something else: a non-identity natural endomorphism for an arbitrary topological space.

First solution

Suppose that we are given for every topological space X a continuous map

$$f_X : X \longrightarrow X$$

that is natural in X. More explicitly, this means that for every continuous map $g : X \to Y$ the following square diagram has to commute:

$$\begin{array}{ccc} X & \xrightarrow{f_X} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f_Y} & Y \end{array}$$

Let X be an arbitrary topological space, and for every element x of X let g_x be the map from $\{*\}$ to X that picks out the element x, i.e.,

$$g_x : \{*\} \longrightarrow X, \quad * \longmapsto x.$$

The map g_x is continuous, whence the diagram

$$\begin{array}{ccc} \{*\} & \xrightarrow{1_{\{*\}}} & \{*\} \\ {\scriptstyle g_x}\downarrow & & \downarrow{\scriptstyle g_x} \\ X & \xrightarrow{f_X} & X \end{array}$$

commutes. This commutativity means that

$$f_X(x) = f_X(g_x(*)) = g_x(*) = x.$$

As these equalities hold for every element x of X, we find that f_X is the identity map on X.

The answer to the initial question is therefore “no”: there is no non-identity solution.

Second solution

The question at hand is whether the identity functor of Top admits a non-trivial endomorphism. Suppose such an endomorphism α were to exist.

Let U be the forgetful functor from Top to Set. The whiskered natural transformation $U\alpha$ from $U1_{\mathsf{Top}} = U$ to $U1_{\mathsf{Top}} = U$ is again non-identity because the functor U is faithful. More explicitly, there exists some object X of Top for which $\alpha_X \neq 1_X$, and then also $(U\alpha)_X = U\alpha_X \neq U1_X = 1_{UX}$.

This shows that every non-identity endomorphism of the identity functor 1_{Top} induces a non-identity endomorphism of the forgetful functor U.

But U is represented by the singleton space $\{*\}$. Therefore, because the Yoneda embedding is full and faithful, endomorphisms of U are in one-to-one correspondence with endomorphisms of $\{*\}$.

But $\{*\}$ admits only the identity endomorphism. Consequently, U admits only the identity endomorphism. Even consequentlier, 1_{Top} admits only the identity endomorphism.

Exercise 2.2.vii

The path functor is represented by the unit interval. It follows from the Yoneda embedding that each automorphism of the path functor is induced by an automorphism of the unit interval.

More explicitly, suppose that we have for every topological space X a reparametrization procedure

$$\alpha_X : \ \mathrm{Path}(X) \longrightarrow \mathrm{Path}(X),$$

and that this procedure is natural in X. Then there exists a homeomorphism r of the unit interval I such that

$$\alpha_X(\gamma) = \gamma r$$

for every path γ in X.

We can further characterize homeomorphisms of the unit interval.

Proposition 2.B. Every homeomorphism of the unit interval is strictly monotone, i.e., strictly increasing or strictly decreasing.

Proof. Let f be a homeomorphism of I. The two endpoints 0 and 1 of I are the only two points in I whose removal does *not* split the interval into two path components. Consequently, f needs to permute these two endpoints. So either $f(0) = 0$ and $f(1) = 1$, or $f(0) = 1$ and $f(1) = 0$. We can switch between the two cases by post-composing f with the flip map $\sigma : \ x \mapsto 1 - x$.

Also, f is strictly decreasing if and only if σf is strictly increasing. It therefore suffices to consider the first case.

We observe that if x and y are two points in I with $x < y$ and $f(x) < f(y)$, then we have $f(x) < f(t) < f(y)$ for every t with $x < t < y$. Indeed, suppose otherwise. Then either $f(t) \leq f(x) < f(y)$ or $f(x) < f(y) \leq f(t)$.

- If $f(t) \leq f(x) < f(y)$, then it follows from the intermediate value theorem that
$$f(x) \in [f(t), f(y)] \subseteq f([t, y])$$
even though x is not contained in the interval $[t, y]$. This contradicts the injectivity of f.
- If $f(x) < f(y) \leq f(t)$, then it follows from the intermediate value theorem that
$$f(y) \in [f(x), f(t)] \subseteq f([x, t])$$
even though y is not contained in the interval $[x, t]$. This contradicts the injectivity of f.

This shows that indeed $f(x) < f(t) < f(y)$.

Let now $x, y \in I$ with $x < y$.

- If $x = 0$, then $0 < y$, therefore $f(y) \neq f(0) = 0$, thus $f(y) > 0 = f(x)$.
- If $x \neq 0$, then $0 < x < y$ and thus $f(0) < f(x) < f(y)$, which entails that $f(x) < f(y)$.

We find in every case that $f(x) < f(y)$. ∎

Lemma 2.C. Every strictly monotone, surjective map from I into itself is a homeomorphism.

Proof. Let f be such a map. We may post-compose the map f with the flip map $x \mapsto 1 - x$ to assume that f is strictly increasing.

The strict monotonicity of f ensures that f is injective. Together with the surjectivity of f, this tells us that f is bijective. The inverse of f is again strictly increasing. It hence suffices to show that under the given conditions, f is continuous. (As swapping the roles of f and f^{-1} will then also show that f^{-1} is continuous.)

We have $0 \leq x$ for every $x \in I$, therefore $f(0) \leq f(x)$ for every $x \in I$, thus $f(0) \leq y$ for every $y \in I$ because f is surjective, and hence $f(0) = 0$. We find in the same way that also $f(1) = 1$.

It follows that the extended map

$$g : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} x & \text{if } x \leq 0, \\ f(x) & \text{if } x \in I, \\ x & \text{if } x \geq 1, \end{cases}$$

is continuous if and only if the original map f is continuous. The map g is again strictly increasing and bijective. We show in the following that g is continuous.

Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. There exists for $y := g(x)$ two points $y_1, y_2 \in \mathbb{R}$ with

$$y - \varepsilon < y_1 < y < y_2 < y + \varepsilon\,.$$

It follows for the points $x_1 := f^{-1}(y_1)$ and $x_2 := f^{-1}(y_2)$ that $x_1 < x < x_2$ because the inverse map f^{-1} is again strictly increasing. There hence exists some $\delta > 0$ with $(x - \delta, x + \delta) \subseteq [x_1, x_2]$. We have $f([x_1, x_2]) \subseteq [f(x_1), f(x_2)]$ because f is increasing, and therefore

$$f((x + \delta, x - \delta)) \subseteq f([x_1, x_2]) \subseteq [f(x_1), f(x_2)] = [y_1, y_2] \subseteq (y - \varepsilon, y + \varepsilon)\,.$$

As $\varepsilon > 0$ was arbitrary, this shows that f is continuous. ∎

Corollary 2.D. The homeomorphisms of the unit interval I are precisely those maps from I into itself that are strictly monotone and surjective.

Equivalently, the homeomorphisms are precisely those maps $f : I \to I$ that are continuous, and either strictly increasing with $f(0) = 0$ and $f(1) = 1$ or strictly decreasing with $f(0) = 1$ and $f(1) = 0$. ∎

2.3 Universal properties and universal elements

Exercise 2.3.i

Given a functor $F : \mathsf{C} \to \mathsf{Set}$ and isomorphism $F \cong \mathsf{C}(c, -)$ for some object c of C, the universal element corresponding to this isomorphism is the element of Fc that corresponds to the element 1_c of $\mathsf{C}(c, c)$.

(i)

Let i be the unique non-identity morphism in the category $\mathbb{2}$. The isomorphism $\alpha : \mathsf{Cat}(\mathbb{2}, -) \Rightarrow \mathrm{mor}$ is given by

$$\alpha_{\mathsf{C}}(F) = Fi\,.$$

The universal element corresponding to this isomorphism is therefore the element $\alpha_{\mathbb{2}}(1_{\mathbb{2}}) = 1_{\mathbb{2}} i = i$ of $\mathrm{mor}\,\mathbb{2}$.

(ii)

The Sierpiński space S is given by the two elements $\top$ and $\bot$ and the three open subsets $\emptyset$, $\{\top\}$ and S. The isomorphism $\alpha : \mathsf{Top}(-, S) \Rightarrow \mathcal{O}$ is given by

$$\alpha_X(\chi) = \chi^{-1}(\top)\,,$$

i.e., by taking the preimage of the open point. The universal element corresponding to this isomorphism is therefore the element $\alpha_S(1_S) = 1_S^{-1}(\top) = \{\top\}$ of $\mathcal{O}(S)$, i.e., the open point of S.

(iii)

We find in the same way as for part (ii) that the universal element is the element $\{\bot\}$ of $\mathcal{C}(S)$, i.e., the closed point of S.

Exercise 2.3.ii

(i)

We observe that for every bilinear map $\beta : \Bbbk \times V \to W$ we have the equality

$$\beta(\lambda, v) = \beta(1, \lambda v)\,.$$

for all $\lambda \in \Bbbk$, $v \in V$. This implies that β is uniquely determined by the linear map

$$V \longrightarrow W\,, \quad v \longmapsto \beta(1, v)\,.$$

Suppose conversely that we are given an arbitrary linear map $g : V \to W$. The map

$$\Bbbk \times V \longrightarrow W\,, \quad (\lambda, v) \longmapsto \lambda g(v)$$

is then bilinear.

The above two constructions are mutually inverse, and give an isomorphism of vector spaces

$$\mathrm{Bil}(\Bbbk, V; W) \cong \mathrm{Vect}_{\Bbbk}(V, W)$$

that is natural in both V and W. It follows that we have the sequence of isomorphisms

$$\mathrm{Vect}_{\Bbbk}(\Bbbk \otimes_{\Bbbk} V, W) \cong \mathrm{Bil}(\Bbbk, V; W) \cong \mathrm{Vect}_{\Bbbk}(V, W)$$

that is natural in both V and W. The isomorphism of functors

$$\mathrm{Vect}_{\Bbbk}(k \otimes_{\Bbbk} V, -) \cong \mathrm{Vect}_{\Bbbk}(V, -)$$

tells us that $k \otimes_{\Bbbk} V \cong V$.

(ii)

We have the sequence isomorphisms

$$\begin{aligned}\mathrm{Vect}_{\Bbbk}(U \otimes_{\Bbbk} (V \otimes_{\Bbbk} W), X) &\cong \mathrm{Bil}(U, V \otimes_{\Bbbk} W; X) \\ &\cong \mathrm{Vect}_{\Bbbk}(U, \mathrm{Vect}_{\Bbbk}(V \otimes_{\Bbbk} W, X)) \\ &\cong \mathrm{Vect}_{\Bbbk}(U, \mathrm{Bil}(V, W; X)) \\ &\cong \mathrm{Vect}_{\Bbbk}(U, \mathrm{Vect}_{\Bbbk}(V, \mathrm{Vect}_{\Bbbk}(W, X))) \\ &\cong \mathrm{Tril}(U, V, W; X)\end{aligned}$$

and similarly

$$\begin{aligned}\mathrm{Vect}_{\Bbbk}((U \otimes_{\Bbbk} V) \otimes_{\Bbbk} W, X) &\cong \mathrm{Bil}(U \otimes_{\Bbbk} V, W; X) \\ &\cong \mathrm{Vect}_{\Bbbk}(U \otimes_{\Bbbk} V, \mathrm{Vect}_{\Bbbk}(W, X)) \\ &\cong \mathrm{Bil}(U, V; \mathrm{Vect}_{\Bbbk}(W, X)) \\ &\cong \mathrm{Vect}_{\Bbbk}(U, \mathrm{Vect}_{\Bbbk}(V, \mathrm{Vect}_{\Bbbk}(W, X))) \\ &\cong \mathrm{Tril}(U, V, W; X)\,.\end{aligned}$$

All these isomorphisms are natural in U, V, W and X. The isomorphism of functors

$$\begin{aligned}\mathrm{Vect}_{\Bbbk}(U \otimes_{\Bbbk} (V \otimes_{\Bbbk} W), -) &\cong \mathrm{Tril}(U, V, W; -) \\ &\cong \mathrm{Vect}_{\Bbbk}((U \otimes_{\Bbbk} V) \otimes_{\Bbbk} W, -)\end{aligned}$$

tells us that $U \otimes_{\Bbbk} (V \otimes_{\Bbbk} W) \cong (U \otimes_{\Bbbk} V) \otimes_{\Bbbk} W$.

Exercise 2.3.iii

The isomorphism

$$\alpha : \mathrm{Set}(-, B^A) \cong \mathrm{Set}(- \times A, B)$$

is explicitly given by

$$\alpha_X(\varphi)(x, a) = \varphi(x)(a) \tag{2.2}$$

for every set X, every function $\varphi : X \to B^A$ and all $(x, a) \in X \times A$.

The universal element ev corresponding to the isomorphism α is given by $\alpha_{B^A}(1_{B^A})$. It is thus an element of $\mathrm{Set}(B^A \times A, B)$, i.e., a map from $B^A \times A$ to B. It follows from the explicit formula (2.2) that this map is given by

$$\mathrm{ev}(f, a) = \alpha_{B^A}(1_{B^A})(f, a) = 1_{B^A}(f)(a) = f(a)$$

for all $(f, a) \in B^A \times A$. The map ev is thus given by evaluation of the first item at the second item.

The Yoneda lemma tells us for the functor $F \coloneqq \mathrm{Set}(- \times A, B)$ that the entire natural transformation $\alpha : \mathrm{Set}(-, B^A) \Rightarrow F$ is uniquely determined by the element ev of $F(B^A)$. More explicitly, $\alpha_X(\varphi) = (F\varphi)(\mathrm{ev})$ for every set X, and therefore

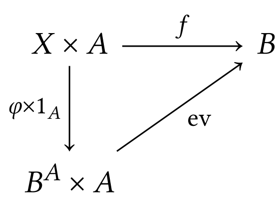
$$\alpha_X(\varphi) = (F\varphi)(\mathrm{ev}) = \mathrm{ev} \cdot (\varphi \times 1_A) .$$

That α is a natural isomorphism means that α_X is bijective for every set X. In other words:

> For every set X and every element f of FX there exists a unique element φ of $\mathrm{Set}(X, B^A)$ with $\alpha_X(\varphi) = f$.

We can expand this condition as follows:

> For every set X and every map $f : X \times A \to B$, there exists a unique map $\varphi : X \to B^A$ such that $f(c, a) = \mathrm{ev}(\varphi(c), a)$ for all $(c, a) \in X \times A$, i.e., such that the following diagram commutes:



2.4 The category of elements

Exercise 2.4.i

Let S be the singleton category consisting of only a single object s and its identity morphism. Let C be the functor from S to Set corresponding to the singleton set $\{*\}$, i.e., the unique functor with $Cs = \{*\}$. The comma category $C \downarrow F$ looks as follows:

- The objects of $C \downarrow F$ are triples of the form (s, c, ξ) consisting of the unique object s of S, an object c of C, and a map f from the set $Cs = \{*\}$ to the set Fc.
- A morphism in $C \downarrow F$ from an object (s, c, ξ) to an object (s, c', ξ') is a pair (f_0, f_1) of morphisms $f_0 : s \to s$ in S and $f : c \to c'$ in C that makes the following diagram commute:

$$\begin{array}{ccc} \{*\} & \xrightarrow{\xi} & Fc \\ {\scriptstyle Cf_0}\downarrow & & \downarrow{\scriptstyle Ff_1} \\ \{*\} & \xrightarrow{\xi'} & Fc' \end{array}$$

The category S has a single object and single morphism. This allows us to simplify the comma category $C \downarrow F$ by replacing it with the following isomorphic category D:

- The objects of D are pairs (c, ξ) consisting of an object c of C and a map $\xi : \{*\} \to Fc$.
- A morphism in D from another object (c, ξ) to an object (c', ξ') is a morphism $f : c \to c'$ in C that makes the following diagram commute:

$$\begin{array}{ccc} \{*\} & \xrightarrow{\xi} & Fc \\ {\scriptstyle 1_{\{*\}}}\downarrow & & \downarrow{\scriptstyle Ff} \\ \{*\} & \xrightarrow{\xi'} & Fc' \end{array} \qquad (2.3)$$

A map $\xi : \{*\} \to Fc$ is the same as an element of Fc, namely the element $\xi(*)$. The commutativity of the square diagram (2.3) is then equivalent to the condition $(Ff)(x) = x'$ for the respective elements $x = \xi(*)$

and $x' = \xi'(*)$. The category D can therefore be replaced by the following isomorphic category E:

- The objects of E are pairs (c, x) consisting of an object c of C and an element x of the associated set Fc.
- A morphism in E from an object (c, x) to another object (c', x') is a morphism $f : c \to c'$ in C such that $(Ff)(x) = x'$.

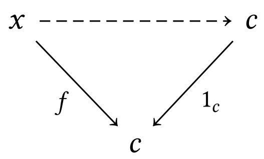
The category E is precisely the category of elements $\int F$, so that overall

$$* \downarrow F \cong \mathsf{D} \cong \mathsf{E} = \textstyle\int F .$$

Exercise 2.4.ii

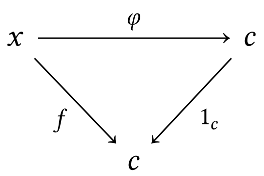
Let C be a category and let c be an object of C.

One terminal object of C/c is given by $(c, 1_c)$. Indeed, for every object (x, f) of C/c there exists a unique morphism from x to c that makes the diagram

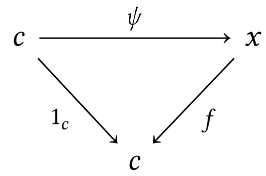


commute, namely the morphism f itself.

It follows that an object (x, f) of C/c is terminal in C/c if and only if it is isomorphic to the object $(c, 1_c)$. This is the case if and only if there exists morphisms $\varphi : (x, f) \to (c, 1_c)$ and $\psi : (c, 1_c) \to (x, f)$ in C/c such that $\psi\varphi = 1_{(c,1_c)}$ and $\varphi\psi = 1_{(x,f)}$. This means that φ and ψ are morphisms in C, namely $\varphi : x \to c$ and $\psi : c \to x$, such that the diagrams



and



commute, and such that $\varphi\psi = 1_c$ and $\psi\varphi = 1_x$ in C. The commutativity of the first diagram is equivalent to the condition $\varphi = f$, and the commutativity of the second diagram is equivalent to the condition $f\psi = 1_c$. We find that φ

must be equal to f and that ψ must be a two-sided inverse to f. Consequently, such morphisms φ and ψ exist if and only if f is an isomorphism in C.

This shows that the terminal objects in C/c are precisely those objects (x, f) for which the morphism f in C is an isomorphism in C. Therefore, roughly speaking, the terminal objects in C/c are the ways in which the object c is isomorphic to another object of C.

Exercise 2.4.iii

Let F be a contravariant functor from a category C to the category Set. We may regard F as a covariant functor G from C^{op} to Set. We then have the sequence of equivalences

$$
\begin{aligned}
& \text{the contravariant functor } F \text{ is representable} \\
\iff & \text{the covariant functor } G \text{ is representable} \\
\iff & \text{the category } \textstyle\int G \text{ admits an initial object} \\
\iff & \text{the category } (\textstyle\int F)^{\mathrm{op}} \text{ admits an initial object} \\
\iff & \text{the category } \textstyle\int F \text{ admits a terminal object}\,.
\end{aligned}
$$

For the second to last equivalence we used that $\int F$ is defined as $(\int G)^{\mathrm{op}}$.

One can refine the above argumentation to get the following result:

> Let F be a contravariant functor from a category C to Set. An element x of a set Fc, where c is some object of C, is a universal element for F if and only if (c, x) is terminal in $\int F$.

Exercise 2.4.iv

Let Ω be the Sierpiński space, consisting of the two elements $\top$ and $\bot$ and three open subsets $\emptyset$, $\{\top\}$ and Ω.

Let $\mathcal{O}$ be the contravariant functor from Top to Set that assigns to each topological space its set of open subsets. We find that

$$
\begin{aligned}
& \left\{
\begin{array}{l}
\text{there exists for every topological space } X \\
\text{and every open subset } U \text{ of } X \\
\text{a unique continuous map } \chi \text{ from } X \text{ to } \Omega \text{ with } U = \chi^{-1}(\{\top\})
\end{array}
\right. \\
\iff & \left\{
\begin{array}{l}
\text{there exists for every object } X \text{ of Top and every element } U \text{ of } \mathcal{O}(X) \\
\text{a unique morphism } \chi\colon X \to \Omega \text{ in Top with } U = \mathcal{O}(\chi)(\{\top\})
\end{array}
\right.
\end{aligned}
$$

We know that the first condition holds true, so the second condition also holds true. But the second condition tells us that the element $\{\top\}$ of $\mathcal{O}(\Omega)$ is a universal element for the functor F. In this sense, the open subset $\{\top\}$ of Ω is the universal open subset. In a weaker sense, this means that Ω is the universal topological space with an open subset.

Exercise 2.4.v

The objects of $\int F$ are pairs $(X, \leq)$ consisting of a set X together with a preorder $\leq$ on X. In other words, the objects of $\int F$ are preordered sets.

A morphism $f : (X, \leq_X) \to (Y, \leq_Y)$ is a map $f : X \to Y$ such that $\leq_X$ is the pullback of $\leq_Y$ along f. In other words, the preorders $\leq_X$ and $\leq_Y$ and the map f need to satisfy the compatibility condition

$$x \leq_X x' \iff f(x) \leq_Y f(x')$$

for all $x, x' \in X$. (This entails that the map f is increasing with respect to the preorders $\leq_X$ and $\leq_Y$.)

We claim that the functor F is not representable.

To prove this claim, we note that for every preorder $\leq$ on a set X we have an associated equivalence relation $\sim$ given by

$$x \sim x' \iff (x \leq x' \text{ and } x' \leq x) .$$

We call the number of equivalence classes of this equivalence relation the **order size** of $\leq$, and denote it by $\operatorname{size}(X, \leq)$, or simply by $\operatorname{size}(X)$.

The order size of a preorder tells how many elements can be at most distinguished by that preorder. Consequently, if a preorder on a set X is the pullback of a preorder on a set Y via a map $f : X \to Y$, then

$$\operatorname{size}(X) \leq \operatorname{size}(Y) .$$

Suppose that the functor F were representable by an object R and a universal element $\leq_R$. This would mean that there exists for every preordered set $(X, \leq_X)$ a unique function $f : X \to R$ such that $\leq_X$ is the pullback of $\leq_R$ along f. This then entails that the order size of *every* preordered set X would be bound by that of the representing object:

$$\operatorname{size}(X) \leq \operatorname{size}(R)$$

for every preordered set X and the representing preordered set R. To show that the functor F is not representable it therefore suffices to show that for any arbitrarily large cardinal number κ there exists a preordered set whose order size at least κ.

We can consider for every set X the discrete preorder on X, for which no two distinct elements are comparable. The order size of X is then its cardinality. As there are sets of arbitrarily large cardinality, we find that there are preordered sets of arbitrarily large order size.

Exercise 2.4.vi

We denote the category of elements of the functor Hom by E, i.e.,

$$\mathsf{E} := \int \mathrm{Hom}\ .$$

The category E looks as follows:

- The objects of E are pairs $((c, d), f)$ consisting of an object (c, d) of $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ and an element f of $\mathrm{Hom}(c, d)$.
- A morphism φ in E from an object $((c, d), f)$ to an object $((c', d'), f')$ is a morphism $\varphi : (c, d) \to (c', d')$ in $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ such that $\mathrm{Hom}(-, -)(\varphi)(f) = f'$.

We can simplify this description of E by unraveling the structure of $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ and of the bifunctor Hom. We then arrive at the following category T:

- The objects of T are triples (c, f, d) consisting of two objects c and d of C and a morphism $f : c \to d$ in C.
- A morphism in T from an object (c, f, d) to another object (c', f', d') is a pair (φ, ψ) of morphisms $\varphi : c' \to c$ and $\psi : d \to d'$ in C such that $\psi f \varphi = f'$, i.e., such that the following square diagram commutes:

$$\begin{array}{ccc} c & \xrightarrow{f} & d \\ {\scriptstyle\psi}\uparrow & & \downarrow{\scriptstyle\varphi} \\ c' & \xrightarrow{f'} & d' \end{array} \qquad (2.4)$$

The objects in T are thus the morphisms in C, and a morphism in T between two morphisms f and f' in C is a commutative diagram of the form (2.4). In this diagram, the morphism f' arises from the morphism f by "twisting" with the two morphisms φ and ψ.

Exercise 2.4.vii

We will use the following observations:

Lemma 2.E. Let C be a category and let (D, Π) and (D′, Π′) be two categories over C, i.e., two objects of CAT/C, for which the functor Π′ is faithful. For every object d of D let Fd be an object of D′, and for every morphism $f : d \to d'$ in D let Ff be a morphism in D′ from Fd to Fd'. If the diagram

$$\begin{array}{ccc} \mathsf{D} & \xrightarrow{F} & \mathsf{D}' \\ & {\scriptstyle\Pi}\searrow \quad \swarrow{\scriptstyle\Pi'} & \\ & \mathsf{C} & \end{array} \tag{2.5}$$

commutes, then F is a functor, and more specifically a morphism from (D, Π) to (D′, Π′) in CAT/C.

Proof. It remains to show the functoriality of F. We hence need to show that

$$F1_d = 1_{Fd} \quad \text{and} \quad F(gf) = Fg \cdot Ff$$

for every object d of D, and for every two composable morphisms $f : d \to d'$ and $g : d' \to d''$ in D. As Π′ is faithful, these two conditions are equivalent to the conditions

$$\Pi' F1_d = \Pi' 1_{Fd} \quad \text{and} \quad \Pi' F(gf) = \Pi'(Fg \cdot Ff)\,.$$

By the functoriality of Π′ we can rewrite these two equations as follows:

$$\Pi' F1_d = 1_{\Pi' Fd} \quad \text{and} \quad \Pi' F(gf) = \Pi' Fg \cdot \Pi' Ff\,.$$

By the commutativity of the diagram (2.5) we can now simplify these two equations to

$$\Pi 1_d = 1_{\Pi d} \quad \text{and} \quad \Pi(gf) = \Pi g \cdot \Pi f\,.$$

These final equations are satisfied by the functoriality of Π. ∎

Let C be a category. For every functor F from C to Set we denote its category of elements by $\int F$, and the forgetful functor from $\int F$ to C by Π_F. The pair $(\int F, \Pi_F)$ is an object of the slice category CAT/C.

Let $F, G : \mathsf{C} \to \mathsf{Set}$ be two functors and let $\alpha : F \Rightarrow G$ be a natural transformation. The natural transformation α induces a functor

$$\textstyle\int\alpha : \int F \longrightarrow \int G$$

as follows:

- Let (c, x) be an object of $\int F$. This means that c is an object of C and x is an element of the set Fc. The component α_c of the transformation α is a map from Fc to Gc, whence

  $$\left(\textstyle\int\alpha\right)(c, x) := (c, \alpha_c(x))$$

  is an object of $\int G$.
- Let $f : (c, x) \to (c', x')$ be a morphism in $\int F$. This means that f is a morphism in C from c to c' with $(Ff)(x) = x'$. We have the following commutative square diagram by the naturality of α:

  $$\begin{array}{ccc} Fc & \xrightarrow{Ff} & Fc' \\ {\scriptstyle\alpha_c}\downarrow & & \downarrow{\scriptstyle\alpha_{c'}} \\ Gc & \xrightarrow[Gf]{} & Gc' \end{array}$$

  The commutativity of this diagram tells us that

  $$(Gf)(\alpha_c(x)) = \alpha_{c'}((Ff)(x)) = \alpha_{c'}(x')\,,$$

  whence f is a morphism from $(c, \alpha_c(x))$ to $(c', \alpha_{c'}(x'))$. In other words, f is a morphism from $(\int\alpha)(c, x)$ to $(\int\alpha)(c', x')$. We therefore define

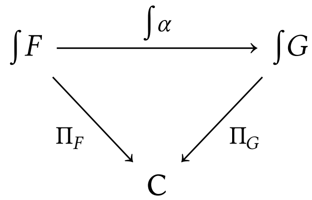
  $$\left(\textstyle\int\alpha\right)\left((c, x) \xrightarrow{f} (c', x')\right) := \left((c, \alpha_c(x)) \xrightarrow{f} (c', \alpha_{c'}(x'))\right).$$

  For simplicity, we will simply write

  $$\left(\textstyle\int\alpha\right)(f) = f\,,$$

  not keeping track of the change in domain and codomain. This greatly improves readability, at the minor cost of some rigour.

- The diagram



(2.6)

  commutes, whence it follows from Lemma 2.E that $\int \alpha$ is a functor from $\int F$ to $\int G$.

  It remains to show that this induced functor $\int \alpha$ is *itself* functorial in α.

- We need to show that $\int 1_F = 1_{(\int F)}$. This holds true because

$$\left(\int 1_F\right)(c, x) = (c, (1_F)_c(x)) = (c, 1_{Fc}(x)) = (c, x),$$

  for every object (c, x) of $\int F$, as well as

$$\left(\int 1_F\right)(f) = f$$

  for every morphism f in $\int F$.

- We need to show that $\int(\beta \cdot \alpha) = \int \beta \cdot \int \alpha$ for every two composable natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$.This holds true because

$$\begin{aligned} \left(\int \beta\right)\left(\int \alpha\right)(c, x) &= \left(\int \beta\right)(c, \alpha_c(x)) \\ &= (c, \beta_c(\alpha_c(x))) \\ &= (c, (\beta \cdot \alpha)_c(x)) \\ &= \left(\int(\beta \cdot \alpha)\right)(c, x) \end{aligned}$$

  for every object (c, x) of $\int F$, as well as

$$\left(\int \beta\right)\left(\int \alpha\right) f = \left(\int \beta\right) f = f = \left(\int(\beta \cdot \alpha)\right) f$$

  for every morphism f in $\int F$.

We have overall extended the construction $\int$ to a functor from the functor category $\mathsf{Set}^{\mathsf{C}}$ to the slice category CAT/C. This entails that isomorphic objects of $\mathsf{Set}^{\mathsf{C}}$ are mapped to isomorphic objects of CAT/C. More explicitly, if $F, G : \mathsf{C} \to \mathsf{Set}$ are two isomorphic functors, then the two categories $\int F$ and $\int G$ are isomorphic over C.

Exercise 2.4.viii

For this exercise we denote the morphisms in $\int F$ as quintuples (c, x, f, d, y) consisting of two objects c and d of C, elements x of Fc and y of Fd, and a morphism $f\colon c \to d$ in C with $(Ff)(x) = y$. Pictorially,

$$Fc \ni x \xrightarrow{Ff} y \in Fd\,.$$

Such a quintuple (c, x, f, d, y) is a morphism from (c, x) to (d, y) in $\int F$.

Let $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)$ be a morphism in $\int F$. We make the following observations:

- That φ is a lift of f along Π is equivalent to the equalities $\varphi_1 = c$, $\varphi_4 = d$ and $\varphi_3 = f$.
- That the domain of φ is (c, x) is equivalent to the two equalities $\varphi_1 = c$ and $\varphi_2 = x$.
- The last entry φ_5 is uniquely determined by the previous entries φ_2 and φ_3 as $\varphi_5 = (F\varphi_3)(\varphi_2)$.

Combining all of these observations, we see that $(c, x, f, d, (Ff)(x))$ is the unique lift of f along Π whose domain is (c, x).

Exercise 2.4.ix

We have the following definition of a discrete right fibration:

> A functor $\Pi\colon \mathsf{E} \to \mathsf{B}$ is a **discrete right fibration** if for every morphism $f\colon c \to d$ in B and every object e in the fibre over d there exists a unique lift of f along Π with codomain e.

Exercise 2.4.x

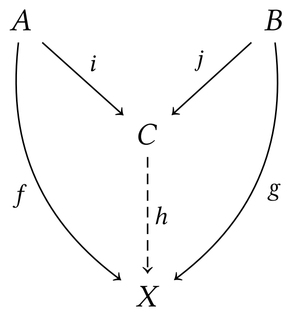
For the functor $\mathsf{C}(A, -) \times \mathsf{C}(B, -)$ to be representable we need the existence of an object C of C such that

$$\mathsf{C}(C, -) \cong \mathsf{C}(A, -) \times \mathsf{C}(B, -)\,.$$

We can also characterize such an isomorphism in terms of its universal element $(i, j) \in \mathsf{C}(A, C) \times \mathsf{C}(B, C)$: we would need an object C of C together with two morphisms

$$i\colon A \longrightarrow C\,, \quad j\colon B \longrightarrow C$$

such that for every object X of C and every two morphisms $f : A \to X$ and $g : B \to X$ there exists a unique morphism $h : C \to X$ with $f = hi$ and $g = hj$:



Chapter 3

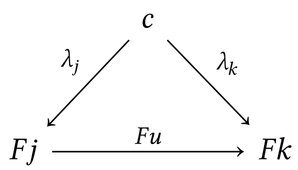
Limits and Colimits

3.1 Limits and colimits as universal cones

Exercise 3.1.i

The action of $\mathrm{Cone}(-, F)$ **on morphisms**

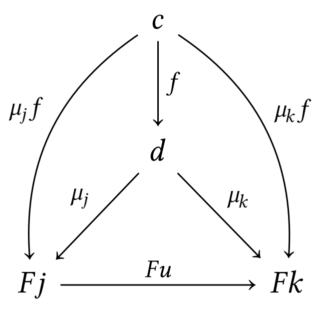
A cone over F with summit c, where c is some object in C, is a family $(\lambda_j)_{j \in \mathsf{J}}$ of morphism $\lambda_j \colon c \to Fj$ subject to the commutativity of the triangular diagram



for every morphism $u \colon j \to k$ in J. For every morphism $f \colon c \to d$ in C we have therefore the induced map

$$f^* \colon \mathrm{Cone}(d, F) \longrightarrow \mathrm{Cone}(c, F), \quad (\mu_j)_{j \in \mathsf{J}} \longmapsto (\mu_j f)_{j \in \mathsf{J}} .$$

This map is well-defined since in the diagram



the outer triangle commutes if the lower inner triangle commutes. (Intuitively speaking, we are pulling back the legs λ along f.)

The action of $\mathrm{Cone}(F, -)$ on morphisms

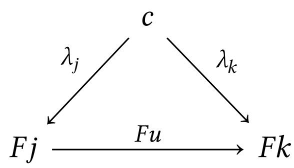
The action of $\mathrm{Cone}(F, -)$ on morphisms is similarly given by

$$f_* : \ \mathrm{Cone}(F, c) \longrightarrow \mathrm{Cone}(F, d), \quad (\lambda_j)_{j \in \mathsf{J}} \longmapsto (f\lambda_j)_{j \in \mathsf{J}}$$

for every morphism $f : \ c \to d$ in C. (Intuitively speaking, we are pushing the legs λ forward along f.)

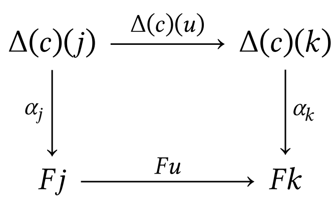
Exercise 3.1.ii

Given an object c of C, an element of the set $\mathrm{Cone}(c, F)$ is a family $(\lambda_j)_{j \in \mathsf{J}}$ of morphisms $\lambda_j : \ c \to Fj$, subject to the commutativity of the triangular diagram



for every morphism $u : \ j \to k$ in J.

An element α of the set $\mathrm{Hom}(\Delta(c), F)$ is a natural transformation from $\Delta(c)$ to F. More explicitly, $\alpha = (\alpha_j)_{j \in J}$ is a family of morphisms $\alpha_j : \ \Delta(c)(j) \to Fj$ such that the square diagram



commutes for every morphism $u : \ j \to k$ in the index category J. But we know that $\Delta(c)(j) = c$ for every object j of J, and that $\Delta(c)(u) = 1_c$ for every morphism u in J. The above square diagram can therefore be simplified as follows:

$$\begin{CD} c @>1_c>> c \\ @V\alpha_jVV @VV\alpha_kV \\ Fj @>Fu>> Fk \end{CD}$$

This square diagram commutes if and only if the following triangular diagram commutes:

$$\begin{array}{ccc} & c & \\ {}^{\alpha_j}\swarrow & & \searrow^{\alpha_k} \\ Fj & \xrightarrow{\;Fu\;} & Fk \end{array}$$

We find that α the family α is a natural transformation from $\Delta(c)$ to F if and only if it is a cone on F with summit c. In other words, we have an equality of sets

$$\mathrm{Hom}(\Delta(c), F) = \mathrm{Cone}(c, F)\,.\text{[1]} \tag{3.1}$$

It remains to check that the equality (3.1) is natural in c. In other words, we need to check that for every morphism $f : c \to d$ in C the following square diagram commutes:

$$\begin{CD} \mathrm{Cone}(d, F) @>f^*>> \mathrm{Cone}(c, F) \\ @| @| \\ \mathrm{Hom}(\Delta(d), F) @>\Delta(f)^*>> \mathrm{Hom}(\Delta(c), F) \end{CD}$$

- The map f^* is given on every cone $(\lambda_j)_j$ by
$$f^*((\lambda_j)_j) = (\lambda_j f)_j\,.$$
- The map $\Delta(f)^*$ is given on every natural transformation $\alpha = (\alpha_j)_j$ by the components
$$(\Delta(f)^*(\alpha))_j = (\alpha \cdot \Delta(f))_j = \alpha_j \cdot \Delta(f)_j = \alpha_j f\,,$$
so that in total
$$\Delta(f)^*((\alpha_j)_j) = (\alpha_j f)_j\,.$$

We see that both maps coincide, as required.

---

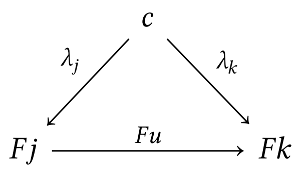
[1]This shows that the two definitions of $\mathrm{Cone}(c, F)$ provided in Definition 3.1.2 coincide.

Exercise 3.1.iii

Cones over F

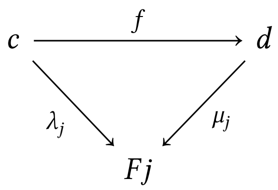
The category Cones(F) of cones over F is defined as a category of elements of the contravariant functor Cone$(-, F)$. This category therefore looks as follows:

- The objects of Cones(F) are pairs (c, λ) consisting of an object c of C and a cone $\lambda = (\lambda_j)_j$ over F with summit c. This means that each λ_j is a morphism from c to Fj, subject to the commutativity of the triangular diagram



  for every morphism $u : j \to k$ in J.

- A morphism in Cones(F) from a cone (c, λ) to a cone (d, μ) is a morphism $f : c \to d$ in C such that the triangular diagram



  commutes for every object j of J. In other words, we have for the induced map

$$f^* : \operatorname{Cone}(d, F) \longrightarrow \operatorname{Cone}(c, F), \quad (\nu_j)_j \longmapsto (\nu_j f)_j$$

  the equality $\lambda = f^*(\mu)$.

The category $\Delta \downarrow F$, on the other hand, looks as follows:

- The objects of $\Delta \downarrow F$ are pairs (c, α) consisting of an object c of C and a natural transformation $\alpha : \Delta(c) \Rightarrow F$.

- A morphism in $\Delta \downarrow F$ from an object (c, α) to an object (d, β) is a morphism $f : c \to d$ in C such that the following triangular diagram commutes:

$$\begin{array}{ccc} \Delta(c) & \xrightarrow{\Delta(f)} & \Delta(d) \\ & {\scriptstyle \alpha}\searrow \qquad \swarrow{\scriptstyle \beta} & \\ & F & \end{array}$$

  That is, we have the equality $\alpha = \Delta(f)^*(\beta)$.

We have already seen in our solution to the previous exercise (Exercise 3.1.ii) that objects of Cones(F) are the same as objects of $\Delta \downarrow F$, as a family $\lambda = (\lambda_j)_j$ is a cone over F with summit c if and only if λ is a natural transformation from $\Delta(c)$ to F.

Let (c, λ) and (d, μ) be two objects of the two categories, and let $f : c \to d$ be a morphism in C. We have also seen in our solution to the previous exercise that the two induced maps

$$f^* : \ \mathrm{Cone}(d, F) \longrightarrow \mathrm{Cone}(c, F)$$

and

$$\Delta(f)^* : \ \mathrm{Hom}(\Delta(d), F) \longrightarrow \mathrm{Hom}(\Delta(c), F)$$

are equal. We have consequently the sequence of equivalences

$$\begin{aligned} & f \text{ is a morphism from } (c, \lambda) \text{ to } (d, \mu) \text{ in } \mathrm{Cones}(F) \\ \iff & \lambda = f^*(\mu) \\ \iff & \lambda = \Delta(f)^*(\mu) \\ \iff & f \text{ is a morphism from } (c, \lambda) \text{ to } (d, \mu) \text{ in } \Delta \downarrow F\,. \end{aligned}$$

This shows that not only are the objects of Cones(F) and $\Delta \downarrow F$ equal, but also their morphisms.

Consequently, the categories Cones(F) and $\Delta \downarrow F$ are equal.

Cocones under F

We can show in the same way that the category of cocones under F is equal to the comma category $F \downarrow \Delta$.

Exercise 3.1.iv

We simply rewrite the first proof in a more elaborate way.

The universal property of a limit cone (ℓ, λ) over F asserts that for every other cone (d, μ) over F, there exists a unique morphism of cones f from (d, μ) to (ℓ, λ). The cone (ℓ', λ') satisfies the same universal property: there exists for every cone (d, μ) on F a unique morphism of cones f' from (d, μ) to (ℓ', λ').

It follows is particular that there exists a unique morphism of cones f from (ℓ, λ) to (ℓ', λ'), as well as a unique morphism of cones f' from (ℓ', λ') to (ℓ, λ). The composite $f' f$ is a morphism of cones from (ℓ, λ) to itself. But by the universal property of (ℓ, λ), there exists precisely one such morphism, and $1_{(\ell,\lambda)}$ is also such a morphism. Consequently,

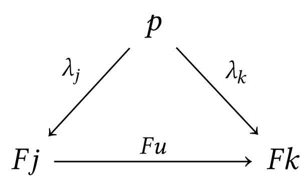
$$f' f = 1_{(\ell,\lambda)} \,.$$

We find in the same way that also $f f' = 1_{(\ell',\lambda')}$.

We have seen that there exists a unique morphism of cones from (ℓ, λ) to (ℓ', λ'), and that this morphism is an isomorphism. This entails that ℓ and ℓ' are isomorphic as objects in C, and that there exists precisely one such isomorphism compatible with the cones λ and λ'.

Exercise 3.1.v

The general definition of a cone over F is as follows:

> A cone over F with summit p, where p is some object in P, is a family $(\lambda_j)_{j \in \mathsf{J}}$ of morphisms $\lambda_j \colon p \to Fj$ with $j \in \mathsf{J}$, subject to the commutativity of the triangular diagram
>
> 
>
>
> for every morphism $u \colon j \to k$ in J.

The category P is a poset, whence every diagram in P commutes. We can therefore simplify the above definition:

> A cone over F with summit p, where p is some object in P, is a family $(\lambda_j)_{j \in \mathsf{J}}$ of morphisms $\lambda_j \colon p \to Fj$ with $j \in \mathsf{J}$.

There exists for every object p of P at most one morphism from p to Fj in P, and this morphism exists if and only if $p \leq Fj$. We therefore find the following:

> Let p be an object of P. There exists a cone on F with summit p if and only if $p \leq Fj$ for every $j \in \mathsf{J}$. This cone is then unique.

Suppose now that p and p' are two summits of cones over F. Every morphism from p to p' is then automatically a morphism of cones, because every diagram in P commutes. Consequently, there exists a morphism of cones from p to p' if and only if there exists a morphism from p to p' in P, which is the case if and only if $p \leq p'$. We thus find the following:

> Let p be an object of P with $p \leq Fj$ for every $j \in \mathsf{J}$, i.e., the summit of a cone over F. The cone determined by p is a limit cone for F if and only if for every other summit p' we have $p' \leq p$.

In other words, a limit cone over F is uniquely determined by its summit, which is precisely a greatest lower bound for the objects Fj with $j \in \mathsf{J}$. The limit of F is thus the infimum of the elements Fj with $j \in \mathsf{J}$.

Dually, the colimit of F is the supremum of the elements Fj with $j \in \mathsf{J}$.

Exercise 3.1.vi

The universal property of the equalizer tells us for every object X of the ambient category C that the map

$$h_* \colon \mathsf{C}(X, E) \longrightarrow \mathsf{C}(X, A), \quad k \longmapsto hk$$

restricts to a bijection

$$\mathsf{C}(X, E) \longrightarrow \{l \in \mathsf{C}(X, A) \mid fl = gl\}.$$

This entails that the map h_* is injective. That this injectivity holds for every object X of C means precisely that the morphism h is a monomorphism.

Exercise 3.1.vii

Let $l, m : Q \to P$ be two morphisms in the ambient category C with $kl = km$. We then also have

$$fhl = gkl = gkm = fhm$$

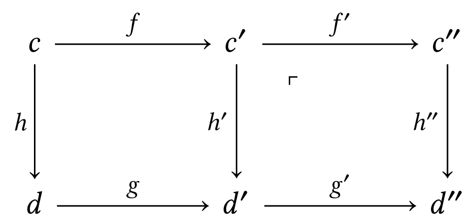
by the commutativity of the pullback square, and thus $hl = hm$ because f is a monomorphism. It follows from the two equalities

$$kl = km\,, \quad hl = hm$$

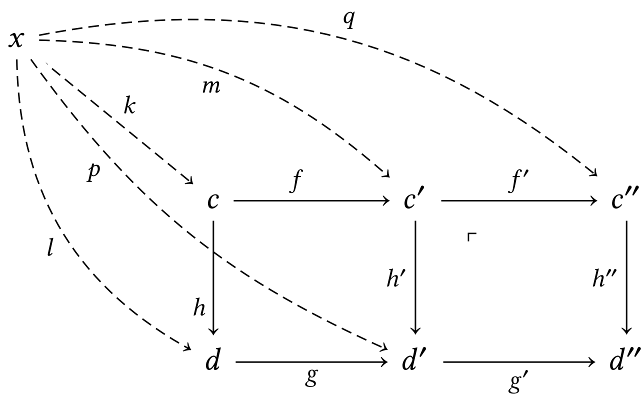
and the universal property of the pullback (P, h, k) that already $l = m$.

Exercise 3.1.viii

We label the objects and morphisms in the given diagram as follows:



We provide a diagram to keep track of the auxiliary morphisms that will be introduced in our argumentations:



Suppose first that the left-hand square diagram is a pullback square. We then have for every object x of C the sequence of bijections

$$
\begin{aligned}
&\{k \mid k \colon x \to c\} \\
\cong &\left\{(l,m) \;\middle|\; \begin{array}{l} l \colon x \to d \text{ and } m \colon x \to c' \\ \text{with } gl = h'm \end{array} \right\} \\
\cong &\left\{(l,p,q) \;\middle|\; \begin{array}{l} l \colon x \to d,\ p \colon x \to d' \text{ and } q \colon x \to c'' \\ \text{with } gl = p \text{ and } g'p = h''q \end{array} \right\} \\
\cong &\left\{(l,q) \;\middle|\; \begin{array}{l} l \colon x \to d \text{ and } q \colon x \to c'' \\ \text{with } g'gl = h''q \end{array} \right\}
\end{aligned}
$$

given by

$$k \longmapsto (hk, fk) \longmapsto (hk, h'fk, f'fk) \longmapsto (hk, f'fk)\,.$$

This bijection tells us that the outer rectangular diagram is a pullback square.

Suppose now conversely that the outer rectangular diagram is a pullback square. We then have for every object x of C the sequence of bijections

$$
\begin{aligned}
&\{k \mid k \colon x \to c\} \\
\cong &\left\{(l,q) \;\middle|\; \begin{array}{l} l \colon x \to d \text{ and } q \colon x \to c'' \\ \text{with } g'gl = h''q \end{array} \right\} \\
\cong &\left\{(l,p,q) \;\middle|\; \begin{array}{l} l \colon x \to d,\ p \colon x \to d' \text{ and } q \colon x \to c'' \\ \text{with } p = gl \text{ and } g'p = h''q \end{array} \right\} \\
\cong &\left\{(l,m) \;\middle|\; \begin{array}{l} l \colon x \to d \text{ and } m \colon x \to c' \\ \text{with } h'm = gl \text{ and } g'h'm = h''f'm \end{array} \right\} \\
= &\left\{(l,m) \;\middle|\; \begin{array}{l} l \colon x \to d \text{ and } m \colon x \to c' \\ \text{with } h'm = gl \end{array} \right\} \text{[2]}
\end{aligned}
$$

given by

$$k \longmapsto (hk, f'fk) \longmapsto (hk, ghk, f'fk) \longmapsto (hk, m)$$

where m is the unique morphism from x to c' with $h'm = ghk$ and $f'm = f'fk$. The morphism fk satisfies these defining equations of m, whence $m = fk$. The above bijection is thus overall given by

$$k \longmapsto (hk, fk)\,.$$

[2] We can drop the condition $g'h'm = h''f'm$ since it follows from the commutativity of the right-hand square.

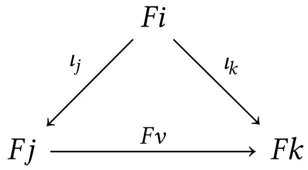
This overall bijection tells us that the left-hand square diagram is a pullback square.

Exercise 3.1.ix

Cones and initial objects

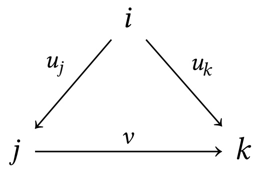
Let i be the initial object of J. There exists for every object j of C a unique morphism u_j from i to j in J. Let $\iota_j := Fu_j$ for every $j \in$ J, which is a morphism in C from Fi to Fj. We claim that ι is a limit cone over F with summit Fi.

We first need to check that ι is a cone over F with summit Fi. To this end, we need to check that for every morphism $v : j \to k$ in J the following triangular diagram commutes:



(3.2)

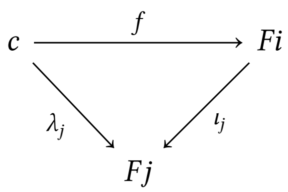
Given that $\iota_j = Fu_j$ and $\iota_k = Fu_k$, this diagram is the image of the diagram



under the functor F. This original diagram in J commutes because there exists *exactly one* morphism from i to k in J. It follows from the functoriality of F that the original diagram (3.2) commutes.

We now need to check that (Fi, ι) is a universal cone over F. We need to show that for every other cone (c, λ) over F there exists a unique morphism of cones f from λ to ι.

- We start with the uniqueness. Let f be a morphism of cones from λ to ι, i.e., a morphism from c to Fi for which the triangular diagram



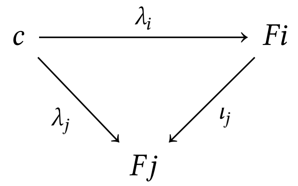
commutes for every object j of J. We consider the case $j = i$. The unique morphism u_i from i to i in J is necessarily the identity morphism 1_i. Consequently,

$$\iota_i = Fu_i = F1_i = 1_{Fi}\,,$$

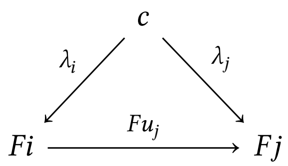
and thus

$$f = 1_{Fi} f = \iota_i f = \lambda_i\,.$$

- For the existence, we now need to check that the morphism λ_i, which goes from c to Fi, is a morphism of cones from λ to ι. To this end, we need to check that the diagram



commutes for every $j \in$ J. This diagram can be rearranged as follows:



The commutativity of this diagram follows from λ being a cone over F.

We have thus proven the following:

> Let J be a category with initial object i, let C be another category and let $F : \mathsf{J} \to \mathsf{C}$ be a diagram of shape J in C. For every object j of J let u_j be the unique morphism from i to j. The family $(Fu_j)_{j\in \mathsf{J}}$ is a limit cone over F with summit Fi.

Cocones and terminal objects

We can dualize the above result:

> Let J be a category with terminal object t, let C be another category and let F be a diagram of shape J in C. For every object j of J let v_j be the unique morphism from j to t. The family $(Fv_j)_{j\in \mathsf{J}}$ is a colimit cone under F with nadir Ft.

Let now α be a successor ordinal. This means that α is the successor of some other ordinal β. The ordinal β is an element of α, and in fact is the unique terminal object of the corresponding category $\mathbb{\alpha}$. Given a diagram F in a category C of shape $\mathbb{\alpha}$, its colimit is thus given by $F\beta$.

Exercise 3.1.x

Pullbacks are only well-defined up to isomorphism. In the given situation, the two cones $(\mathbb{Z}, a, b)$ and $(\mathbb{Z}, -a, -b)$ are isomorphic via the isomorphism of cones

$$\mathbb{Z} \longrightarrow \mathbb{Z}, \quad x \longmapsto -x .$$

Thus, $(\mathbb{Z}, a, b)$ is a pullback for the given diagram if and only if $(\mathbb{Z}, -a, -b)$ is one.

Exercise 3.1.xi

Let C be a category such that for any two objects c and d of C there exists a morphism from c to d in C. Let $(c_j)_{j \in J}$ be a family of objects in C that admits a coproduct $\coprod_{j \in J} c_j$, with structure morphisms $i_k : c_k \to \coprod_{j \in J} c_j$ for $k \in J$.

Let us fix an index $k \in J$. For every index $l \in J$ there exists some morphism r_l from c_l to c_k in C, and we may choose r_k as 1_{c_k}. There exists by assumption a unique morphism r from $\coprod_{j \in J} c_j$ to c_k with $r i_l = r_l$ for every index $l \in J$. This entails that

$$r i_k = r_k = 1_{c_k} ,$$

which tells us that i_k is a split monomorphism.

Exercise 3.1.xii

This exercise is too hard for me.

Exercise 3.1.xiii

The coproduct of finitely many commutative rings $A_1, \dots, A_n$ is their tensor product $A_1 \otimes \cdots \otimes A_n$ over $\mathbb{Z}$ together with the canonical homomorphisms of rings

$$A_i \longrightarrow A_1 \otimes \cdots \otimes A_n , \quad a \longmapsto 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1 .$$

3.2 Limits in the category of sets

Exercise 3.2.i

We abbreviate the sets $\operatorname{mor} C$ and $\operatorname{ob} C$ as M and O respectively. We have thus two sets M and O and the functions

$$\operatorname{dom}, \operatorname{codom} : M \longrightarrow O, \quad \operatorname{id} : O \longrightarrow M.$$

The domain of the composition function is the set

$$\{(f, g) \in M \times M \mid \operatorname{codom} f = \operatorname{dom} g\},$$

which can be described as the pullback of the following diagram:

$$\begin{array}{ccc} & & M \\ & & \big\downarrow{\scriptstyle \operatorname{codom}} \\ M & \xrightarrow[\operatorname{dom}]{} & O \end{array}$$

We thus denote this domain by $M \times_O M$, so that

$$\operatorname{comp} : M \times_O M \longrightarrow M, \quad (f, g) \longmapsto gf$$

is the composition function.

We need to express in terms of diagrams the associativity of comp and that identities act as neutral elements with respect to decomposition.

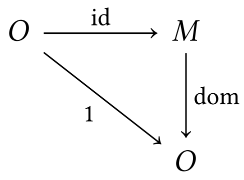
- To express the associativity of composition we need to introduce the triple-pullback

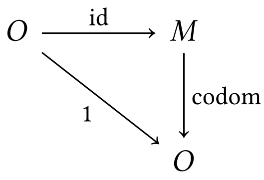
$$M \times_O M \times_O M := \left\{ (f, g, h) \in M \times M \times M \;\middle|\; \begin{array}{l} \operatorname{codom} f = \operatorname{dom} g, \\ \operatorname{codom} g = \operatorname{dom} h \end{array} \right\}.$$

  The associativity of comp can be expressed by the commutativity of the following diagram:

$$\begin{array}{ccc} M \times_O M \times_O M & \xrightarrow{1 \times \operatorname{comp}} & M \times_O M \\ {\scriptstyle \operatorname{comp} \times 1}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{comp}} \\ M \times_O M & \xrightarrow[\operatorname{comp}]{} & M \end{array}$$

- We need for every object x that x is both the domain and the codomain of 1_x: This means that the following two diagrams have to commute:





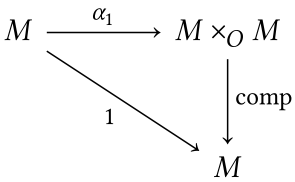
  This then ensures that we have well-defined maps

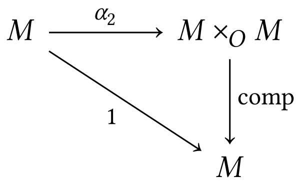
  $$\alpha_1 : M \longrightarrow M \times_O M, \quad f \longmapsto (1_{\operatorname{dom} f}, f)$$

  and

  $$\alpha_2 : M \longrightarrow M \times_O M, \quad f \longmapsto (f, 1_{\operatorname{codom} f}).$$

  We need for the following two diagrams to commute:





Exercise 3.2.ii

In the proof of Theorem 3.2.13 we considered the condition

$$(Ff)(\lambda_{\operatorname{dom} f}) = \lambda_{\operatorname{codom} f}$$

for *every* morphism f in Set. But this condition is trivially satisfied whenever $f = 1_x$ for some object x, since then

$$(Ff)(\lambda_{\operatorname{dom} f}) = (F1_x)(\lambda_{\operatorname{dom} 1_x}) = 1_{Fx}(\lambda_x) = \lambda_x = \lambda_{\operatorname{codom} 1_x} = \lambda_{\operatorname{codom} f}.$$

In the product $\prod_{f \in \operatorname{mor} J} F(\operatorname{codom} f)$ we can therefore leave out all those factors coming from identity morphisms.

Exercise 3.2.iii

We can describe $\mathrm{Sq}(f, g)$ more explicitly as

$$\mathrm{Sq}(f, g) = \{(h, k) \mid h : a \to c,\ k : b \to d,\ gh = kf\}.$$

In other words, $\mathrm{Sq}(f, g)$ is the pullback of the following diagram:

$$\begin{array}{ccc} & & \mathsf{C}(a, c) \\ & & \big\downarrow{\scriptstyle g_*} \\ \mathsf{C}(b, d) & \xrightarrow[f^*]{} & \mathsf{C}(a, d) \end{array}$$

Exercise 3.2.iv

A natural transformation $\alpha : F \Rightarrow G$ is a family $\alpha = (\alpha_j)_{j \in \mathsf{J}}$ such that for every morphism $f : j \to k$ in J the following diagram commutes:

$$\begin{array}{ccc} Fj & \xrightarrow{Ff} & Fk \\ {\scriptstyle \alpha_j}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_k} \\ Gj & \xrightarrow{Gf} & Gk \end{array}$$

In other words, we need the equality

$$\alpha_k \cdot Ff = Gf \cdot \alpha_j$$

for every morphism $f : j \to k$ in C. We can therefore describe $\mathrm{Hom}(F, G)$ as the equalizer

$$\mathrm{Hom}(F, G) \rightarrowtail \prod_{j \in \mathsf{J}} \mathsf{C}(Fj, Gj) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \prod_{\substack{f : j \to k \\ \text{in } \mathsf{J}}} \mathsf{C}(Fj, Gk)$$

where the two maps φ and ψ are given by

$$\varphi(\alpha)_f = \alpha_k \cdot Ff, \quad \psi(\alpha)_f = Gf \cdot \alpha_j,$$

for every morphism $f : j \to k$ in C.

Exercise 3.2.v

We already used this construction in the previous exercise.

Exercise 3.2.vi

The category $\int F$ looks as follows:

- The objects of $\int F$ are pairs (j, x) consisting of an object j of J and an element x of Fj.
- A morphism from (j, x) to (k, y) in $\int F$ is a morphism $f : j \to k$ in J with $y = (Ff)(x)$.

A section Σ to the canonical projection functor $\Pi : \int F \to \mathsf{J}$ therefore looks as follows:

- To every object j of J we associate an object (j, x_j) of $\int F$, with x_j an element of the set Fj.
- To every morphism $f : j \to k$ in J we associate f regarded as a morphism from (j, x_j) to (k, x_k). This means precisely that we have $(Ff)(x_j) = x_k$.

The functor conditions $\Sigma(1_j) = 1_{\Sigma j}$ and $\Sigma(gf) = \Sigma g \cdot \Sigma f$ are then automatically satisfied.

We hence see that a section of Π is the same as a choice of elements $x_j \in Fj$ for $j \in \mathsf{J}$ that is consistent in the sense that $(Ff)(x_j) = x_k$ for every morphism $f : j \to k$ in J. But these conditions mean precisely that the family $(x_j)_{j \in \mathsf{J}}$ defines an element of $\lim F$. We hence see that sections of Π correspond one-to-one to elements of $\lim F$.

This observation allows us to *define* $\lim F$ as the set of sections of Π. The legs $\lambda : \lim F \Rightarrow F$ can then be described as follows: if Σ is a section of Π, and thus an element of $\lim F$, then $\Sigma j = (j, \lambda_j(\Sigma))$. In other words, $\lambda_j(\Sigma)$ is the projection of Σj onto its second coordinate.

3.3 Preservation, reflection, and creation of limits and colimits

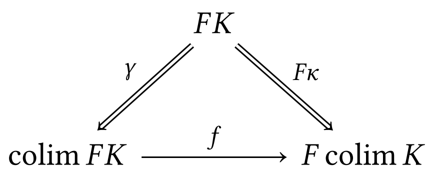
Exercise 3.3.i

(i)

We have in C the colimit cone $\kappa\colon K \Rightarrow \operatorname{colim} K$ under K. Applying the functor F to the diagram K and the cone $\kappa\colon K \to \operatorname{colim} K$ under it yields the cone

$$F\kappa\colon FK \Longrightarrow F \operatorname{colim} K$$

under FK in D. The colimit cone $\gamma\colon FK \Rightarrow \operatorname{colim} FK$ is the initial cone under FK, whence there exists a unique morphism of cones f as depicted:



(ii)

Suppose first that F preserves colimits. The cone $F\kappa\colon FK \Rightarrow F \operatorname{colim} K$ is then a colimit cone under FK. The morphism f is thus an isomorphism of cones by the uniqueness of colimits.

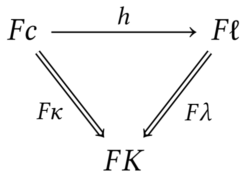
Suppose now that the morphism f is an isomorphism in D. It is then an isomorphism of cones from $\gamma\colon FK \Rightarrow \operatorname{colim} FK$ to $F\kappa\colon FK \Rightarrow F \operatorname{colim} K$. As $\gamma\colon FK \Rightarrow \operatorname{colim} FK$ is initial in the category of cones under FK, it follows that the isomorphic cone $F\kappa\colon FK \Rightarrow F \operatorname{colim} K$ is also initial. In other words, it is again a colimit cone. This shows that F preserves colimits.

Exercise 3.3.ii

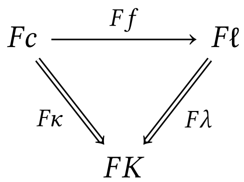
We consider for simplicity only limits. Colimits can be dealt with in the same way.

Let $F\colon \mathsf{C} \to \mathsf{D}$ be a full and faithful functor. Let $K\colon \mathsf{J} \to \mathsf{C}$ be a diagram in C and let $\lambda\colon \ell \Rightarrow K$ be a cone over K. Suppose that the induced cone $F\lambda\colon F\ell \Rightarrow FK$ is a limit cone in D. We need to show that $\lambda\colon \ell \Rightarrow K$ was a limit cone to begin with. In other words, we need to show that for every cone $\kappa : c \Rightarrow K$ in C there exists a unique morphism of cones from $\kappa : c \Rightarrow K$ to $\lambda : \ell \Rightarrow K$.

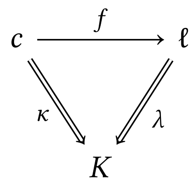
We start with the existence. To this end we consider in D the induced cone $F\kappa : Fc \Rightarrow FK$. There exists a unique morphism of cones h as depicted below because $F\lambda : F\ell \Rightarrow FK$ is a limit cone over FK:



There exists a unique morphism $f : c \to \ell$ in C with $h = Ff$ because F is both full and faithful. We have therefore the following commutative diagram in D:



It follows from the faithfulness of F that the original diagram



in C commutes. This tells us that f is a morphism of cones from $\kappa : c \Rightarrow K$ to $\lambda : \ell \Rightarrow K$.

We now show the uniqueness. To this end let g be any morphism of cones from $\kappa : c \Rightarrow K$ to $\lambda : \ell \Rightarrow K$ in C. It follows that Fg is a morphism of cones from $F\kappa : Fc \to FK$ to $F\lambda : F\ell \to FK$ in D. Therefore, $Fg = h$ by the uniqueness of h. This means that $Fg = Ff$, and thus $g = f$ because F is faithful.

Bibliography

[MSE15] Servaes. *Taking the automorphism group of a group is not functorial.* August 12, 2015. URL: https://math.stackexchange.com/a/1394329 (visited on November 24, 2023).

[MSE18a] Lukas Heger. *Does ring epimorphism induce reversed fully faithful functors in the module categories?* May 28, 2018. URL: https://math.stackexchange.com/a/2798664 (visited on December 2, 2023).

[MSE18b] Nex. *Riehl's Category Theory in Context - Exercise 1.3.ix.* August 7, 2018. URL: https://math.stackexchange.com/a/2874720 (visited on November 23, 2023).

[MSE19] Qiaochu Yuan. *Is every group the unit group of some ring?* September 24, 2019. URL: https://math.stackexchange.com/a/3367453 (visited on June 16, 2023).

[MSE21] John Baez. *When is the ring homomorphism $\mathbb{Z} \to R$ an epimorphism?* September 15, 2021. URL: https://math.stackexchange.com/a/4251470 (visited on December 2, 2023).