Personal Solutions to

Basic Category Theory

by Tom Leinster

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August 9, 2023

Available online at
https://gitlab.com/cionx/solutions-basic-category-theory-leinster.

Preface

The following document contains my personal solutions to the exercises in Tom Leinster's *Basic Category Theory* [Lei14]. There is no guarantee that these solutions are correct or complete.

Similar collections of solutions can be found at [pos21] and [Wei19].

Contents

Chapter 0

Introduction

Exercise 0.10

For every topological space X, every set-theoretic map from X to $I(S)$ is continuous. This can also be formulated as follows:

Let i be the set-theoretic map from S to $I(S)$ given by $i(s) := s$ for every element s of S. There exists for any set-theoretic map f from X to S a unique continuous map $\overline{f}$ from X to $I(S)$ with $\overline{f} = i \circ f$, i.e., such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, dashed, "\overline{f}"] & \\ S \arrow[r, "i"'] & I(S) \end{tikzcd}$$

Exercise 0.11

Let K be another group and let φ be a homomorphism of groups from K to H such that the composite $\theta \circ \varphi$ is the trivial homomorphism of groups. Then there exists a unique homomorphism of groups ψ from K to $\ker(\theta)$ with $\varphi = \iota \circ \psi$, i.e., such that the following diagram commutes:

$$\begin{tikzcd} K \arrow[dr, "\varphi"] \arrow[drr, bend left, "1"] \arrow[dd, dashed, "\psi"'] & & \\ & G \arrow[r, "\theta"] & H \\ \ker(\theta) \arrow[ur, "\iota"] \arrow[urr, bend right, "1"'] & & \end{tikzcd}$$

Exercise 0.12

We recall the following two statement from point-set topology and from naive set theory respectively.

Proposition 0.A. Let X be a topological space and let U and V be two open subsets of X such that $X = U \cup V$. Let Y be another topological space and let f be a set-theoretic map from X to Y. The map f is continuous if and only if both restrictions $f|_U$ and $f|_V$ are continuous.

Proposition 0.B. Let X be a set and let U and V be two subsets of X such that $X = U \cup V$. Let Y be another set and let

$$f : U \longrightarrow Y \quad \text{and} \quad g : V \longrightarrow Y$$

be two maps that agree on the intersection $U \cap V$, i.e., such that

$$f|_{U \cap V} = g|_{U \cap V} .$$

Then there exists a unique map h from X to Y with both $h|_U = f$ and $h|_V = g$.

We consider now the situation of diagram (0.3) from the book. The commutativity of the outer diagram

$$\begin{tikzcd}
U \cap V \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & U \arrow[dd, bend left, "f"] \\
V \arrow[dr, bend right, "g"'] & \\
& Y
\end{tikzcd}$$

means precisely that the functions f and g agree on the intersection $U \cap V$. It follows from Proposition 0.B that there exists a unique set-theoretic map h from X to Y with $h|_U = f$ and $h|_V = g$. This means precisely that the map h

makes the diagram

$$\begin{tikzcd}
U \cap V \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & U \arrow[d, "j'"'] \arrow[ddr, bend left, "f"] & \\
V \arrow[r, "i'"] \arrow[drr, bend right, "g"'] & X \arrow[dr, dashed, "h"] & \\
& & Y
\end{tikzcd}$$

commute. It follows from Proposition 0.A that this map h is again continuous.

Exercise 0.13

(a)

Let ϕ be a homomorphism of rings from $\mathbb{Z}[x]$ to R. Let y be the image of the variable x under ϕ, i.e., let $y := \phi(x)$. For any element p of $\mathbb{Z}[x]$, which is of the form $p = \sum_{n \geq 0} a_n x^n$ for suitable coefficients a_n, we have

$$\phi(p) = \phi\left(\sum_{n \geq 0} a_n x^n\right) = \sum_{n \geq 0} a_n \phi(x)^n = \sum_{n \geq 0} a_n y^n .$$

This shows that the homomorphism ϕ is uniquely determined by its action on the element x of $\mathbb{Z}[x]$. This in turn shows the desired uniqueness.

Let now y be an arbitrary element of R. The map

$$\phi \colon \mathbb{Z}[x] \longrightarrow R , \quad \sum_{n \geq 0} a_n x^n \longmapsto \sum_{n \geq 0} a_n y^n$$

is a homomorphism of rings that maps the variable x onto the element y. This shows the desired existence.

(b)

There exists by the previous part of the exercise a unique homomorphism of rings φ from $\mathbb{Z}[x]$ to A with $\varphi(x) = a$, and there exists similarly a unique homomorphism of rings ψ from A to $\mathbb{Z}[x]$ with $\psi(a) = x$.

There exists by the previous part of the exercise a unique homomorphism of rings from $\mathbb{Z}[x]$ to $\mathbb{Z}[x]$ that maps the element x onto itself. But both $1_{\mathbb{Z}[x]}$ and the composite $\psi \circ \varphi$ are homomorphisms of rings that satisfy this condition. It follows that $\psi \circ \varphi = 1_{\mathbb{Z}[x]}$. We find in the same way that also $\varphi \circ \psi = 1_A$.

We have shown that the two homomorphisms φ and ψ are mutually inverse. This means that φ is an isomorphism whose inverse is given by the homomorphism ψ.

Exercise 0.14

(a)

Let P be the vector space $X \times Y$, let p_1 be the projection map from P to X into the first coordinate, and let p_2 be the projection map from P to Y into the second coordinate.

For every pair (f_1, f_2) of set-theoretic maps

$$f_1 \colon V \longrightarrow X \,, \quad f_2 \colon V \longrightarrow Y \,,$$

there exists a unique set-theoretic map

$$f \colon V \longrightarrow X \times Y$$

such that both $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$, namely the map

$$f \colon V \longrightarrow P \,, \quad v \longmapsto (f_1(v), f_2(v)) \,.$$

The map f is linear if and only if it is linear in both coordinates. In other words, f is linear if and only if both f_1 and f_2 are linear. It follows that the cone (P, p_1, p_2) satisfies the desired universal property.

(b)

By assumption on the cone (P', p'_1, p'_2), there exists a unique linear map i from P to P' such that both

$$p'_1 \circ i = p_1 \quad \text{and} \quad p'_2 \circ i = p_2 \,.$$

There exists similarly a unique linear map j from P' to P such that both

$$p_1 \circ j = p'_1 \quad \text{and} \quad p_2 \circ j = p'_2 \,.$$

By the universal property of the cone (P, p_1, p_2), there exists a unique linear map f from P to P such that both $p_1 \circ f = p_1$ and $p_2 \circ f = p_2$. However, both 1_P and $j \circ i$ satisfy this defining condition of the map f. We have therefore both $f = 1_P$ and $f = j \circ i$, and thus $j \circ i = 1_P$.

We find in the same way that also $i \circ j = 1_{P'}$.

This shows that the linear maps i and j are mutually inverse isomorphisms of vector spaces.

(c)

Let Q be the vector space $X \oplus Y$, and let q_1 and q_2 be the two linear maps

$$q_1 : X \longrightarrow Q, \quad x \longmapsto (x, 0)$$

and

$$q_2 : Y \longrightarrow Q, \quad y \longmapsto (0, y).$$

Let (V, f_1, f_2) be a cocone. We show in the following that there exists a unique linear map f from Q to V with both $f \circ q_1 = f_1$ and $f \circ q_2 = f_2$.

To show the required uniqueness, let f be such a linear map. Then

$$\begin{aligned} f((x,y)) &= f((x,0) + (0,y)) \\ &= f((x,0)) + f((0,y)) \\ &= f(q_1(x)) + f(q_2(y)) \\ &= (f \circ q_1)(x) + (f \circ q_2)(y) \\ &= f_1(x) + f_2(y) \end{aligned}$$

for all $x \in X$, $y \in Y$. This shows that the linear map f is uniquely determined by the composites f_1 and f_2, which shows the desired uniqueness of f.

To show the existence of the linear map f, we define it as

$$f : Q \longrightarrow V, \quad (x, y) \longmapsto f_1(x) + f_2(y).$$

The map f is linear since it can be expressed as

$$f = f_1 \circ p_1 + f_2 \circ p_2,$$

with p_1, p_2, f_1 and f_2 being linear. The map f also satisfies both

$$(f \circ q_1)(x) = f(q_1(x)) = f((x,0)) = f_1(x) + f_2(0) = f_1(x) + 0 = f_1(x)$$

and

$$(f \circ q_2)(y) = f(q_2(y)) = f((0, y)) = f_1(0) + f_2(y) = 0 + f_2(y) = f_2(y) \,.$$

This shows the desired existence of the linear map f.

(d)

Let (Q', q_1', q_2') be a cocone such that for every cocone (V, f_1, f_2) there exists a unique linear map f from Q' to V with $f \circ q_1' = f_1$ and $f \circ q_2' = f_2$.

It follows from the previous part of this exercise that there exists a unique linear map i from Q to Q' such that $i \circ q_1 = q_1'$ and $i \circ q_2 = q_2'$. It follows from the above assumption on (Q', q_1', q_2') that there exists a unique linear map j from Q' to Q such that $j \circ q_1' = q_1$ and $j \circ q_2' = q_2$.

There exists by the previous part of this exercise a unique linear map f from Q to Q such that $f \circ q_1 = q_1$ and $f \circ q_2 = q_2$. Both 1_Q end $j \circ i$ satisfy this defining property of f, which shows that $j \circ i = 1_Q$. We find in the same way that also $i \circ j = 1_{Q'}$.

This shows that the linear maps i and j are mutually inverse isomorphisms of vector spaces.

Chapter 1

Categories, functors and natural transformations

1.1 Categories

Exercise 1.1.12

- The category **Ab** of abelian groups: the class of objects of **Ab** is the class of abelian groups; the morphisms in **Ab** are the homomorphisms of groups between abelian groups; composition of morphisms in **Ab** is given by the usual composition of functions.
- For any ring R, the category R-**Mod** of left R-modules: the class of objects of R-**Mod** is the class of left R-modules; the morphisms in R-**Mod** are the homomorphisms of R-modules; composition of morphisms in R-**Mod** is given by the usual composition of functions.
- The category **Poset** of partially ordered sets: the class of objects of **Poset** is the class of partially ordered sets; the morphisms in **Poset** are the order-preserving maps; composition of morphisms in **Poset** is given by the usual composition of functions.
- The category **Mfld** of smooth manifolds: the class of objects of **Mfld** is the class of smooth (real) manifolds; the morphisms in **Mfld** are the smooth maps between smooth manifolds; composition of morphisms in **Mfld** is given by the usual composition of functions.
- For any field $\Bbbk$, the category $\mathbf{Var}_{\Bbbk}$ of varieties over $\Bbbk$: the class of objects of $\mathbf{Var}_{\Bbbk}$ is the class of varieties over $\Bbbk$; the morphisms in $\mathbf{Var}_{\Bbbk}$ are the regular maps; composition of morphisms in $\mathbf{Var}_{\Bbbk}$ is given by the usual composition of functions.

Exercise 1.1.13

Let g be a left-sided inverse to f and let h be a right-sided inverse to f. Then

$$g = g \circ 1_B = g \circ f \circ h = 1_A \circ h = h\,.$$

Exercise 1.1.14

The composite of two morphisms

$$(f, g) \colon (A, B) \longrightarrow (A', B')\,, \quad (f', g') \colon (A', B') \longrightarrow (A'', B'')$$

in $\mathscr{A} \times \mathscr{B}$ is given by

$$(f', g') \circ (f, g) := (f' \circ f, g' \circ g)\,.$$

The identity morphism of an object (A, B) of $\mathscr{A} \times \mathscr{B}$ is then given by $(1_A, 1_B)$, i.e.,

$$1_{(A,B)} = (1_A, 1_B)\,.$$

Exercise 1.1.15

For any two continuous maps f and g from a topological space X to another topological space Y we write $f \simeq g$ to mean that f and g are homotopic.

The composition of morphisms in **Top** descends do a composition of morphisms in **Toph** thanks to the following observation:

> *Let X, Y and Z be topological spaces. Let*
>
> $$f, f' \colon X \longrightarrow Y\,, \quad g, g' \colon Y \longrightarrow Z$$
>
> *be continuous maps. If $f \simeq f'$ and $g \simeq g'$, then also $g \circ f \simeq g' \circ f'$.*

For any topological space X, the identity morphism of X in **Toph** is given by the homotopy class of the identity map on X. It follows that two topological spaces X and Y are isomorphic in **Toph** if and only if there exist continuous maps

$$f \colon X \longrightarrow Y\,, \quad g \colon Y \longrightarrow X$$

such that both $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. In other words, two topological spaces become isomorphic in **Toph** if and only if they are homotopy equivalent. (More explicitly, a continuous map becomes an isomorphism in **Toph** if and only if it is a homotopy equivalence.)

1.2 Functors

Exercise 1.2.20

- For every topological space X let $\pi_0(X)$ be the set of path components of X. Every continuous map
$$f\colon\ X \longrightarrow Y$$
induces a map
$$\pi_0(f)\colon\ \pi_0(X) \longrightarrow \pi_0(Y)\,, \quad [x] \longmapsto [f(x)]\,.$$
This construction defines a functor π_0 from **Top** to **Set**.
- For every $\Bbbk$-vector space V let $\mathrm{T}(V)$ be the tensor algebra over V. For every two $\Bbbk$-vector spaces V and W and every linear map f from V to W, let $\mathrm{T}(f)$ be the induced homomorphism of $\Bbbk$-algebras from $\mathrm{T}(V)$ to $\mathrm{T}(W)$, which is given by
$$\mathrm{T}(f)(v_1 \otimes \cdots \otimes v_n) = f(v_1) \otimes \cdots \otimes f(v_n)$$
for every $n \geq 0$ and all $v_1, \dots, v_n \in V$. This construction defines a functor T from $\mathbf{Vect}_{\Bbbk}$ to $\mathbf{Alg}_{\Bbbk}$.
  We have similarly functors S and $\bigwedge$ from $\mathbf{Vect}_{\Bbbk}$ to $\mathbf{Alg}_{\Bbbk}$ that assign to every vector space its symmetric algebra and its exterior algebra respectively.
- For every group G let G^{ab} be the abelianization of G. For every two groups G and H and every homomorphism of groups φ from G to H, let φ^{ab} be the induced homomorphism of groups from G^{ab} to H^{ab} given by
$$\varphi^{\mathrm{ab}}([x]) = [\varphi(x)]$$
for every $x \in G$. This construction defines a functor $(-)^{\mathrm{ab}}$ from **Grp** to **Ab**.
- For every group G let $C(G)$ be its set of conjugacy classes. Every homomorphism of groups
$$\varphi\colon\ G \longrightarrow H$$
induces a map
$$C(\varphi)\colon\ C(G) \longrightarrow C(H)\,, \quad [g] \longmapsto [\varphi(g)]\,.$$
This mapping C is a functor from **Grp** to **Set**.

Exercise 1.2.21

Proposition 1.A. Let F be a functor from a category $\mathscr{A}$ to a category $\mathscr{B}$. Let f be an isomorphism in $\mathscr{A}$. Then $F(f)$ is an isomorphism in $\mathscr{B}$.

Proof. Let A be the domain of f, and let A' be its codomain. By applying the functor F to these equations

$$f \circ f^{-1} = 1_{A'}\,, \quad f^{-1} \circ f = 1_A\,,$$

we arrive at the equations

$$F(f) \circ F(f^{-1}) = 1_{F(A')} \quad F(f^{-1}) \circ F(f) = 1_{F(A)}\,.$$

These equations show that the morphism $F(f)$ is again an isomorphism, with inverse given by $F(f^{-1})$.[1] ∎

There exists by assumption an isomorphism f from A to A' in $\mathscr{A}$. It follows from Proposition 1.A that $F(f)$ is an isomorphism from $F(A)$ to $F(A')$ in $\mathscr{B}$. The existence of such an isomorphism shown that the objects $F(A)$ and $F(A')$ are again isomorphic.

Exercise 1.2.22

A functor F from $\mathscr{A}$ to $\mathscr{B}$ consists of the following data:

- A map f from A to B that gives us the action of F on the objects on $\mathscr{A}$.
- For every two elements a and a' of A and every morphism i from a to a' in $\mathscr{A}$ a morphism $F(i)$ from $F(a)$ to $F(a')$ in $\mathscr{B}$, such that the following two properties hold:
  1. $F(1_a) = 1_{f(a)}$ for every element a of $\mathscr{A}$.
  2. For every two composable morphisms

     $$i : a \longrightarrow a'\,, \quad j : a' \longrightarrow a''$$

     in $\mathscr{A}$, the equality $F(j \circ i) = F(j) \circ F(i)$ holds.

In the category $\mathscr{B}$, any two morphisms with the same domain and the same codomain are automatically equal. This has the following two consequences:

[1] In other words, $F(f^{-1}) = F(f)^{-1}$.

- The two conditions 1 and 2 are both automatically satisfied.
- The action of F on morphisms is uniquely determined by the action of F on objects, and thus by the map f.

The data of a functor F from $\mathscr{A}$ to $\mathscr{B}$ does therefore only consist of a function f from A to B satisfying the following condition:

> There exist a morphism from $f(a)$ to $f(a')$ in $\mathscr{B}$ for every two elements a and a' of A for which there exist a morphism from a in a' in $\mathscr{A}$.

In other words, the function f needs to satisfy the condition $f(a) \leq f(a')$ for every two elements a and a' of A with $a \leq a'$. But this is precisely what it means for f to be order-preserving.

Exercise 1.2.23

(a)

The group G^{op} is the opposite group of G: to every element g of G there is an associated element g^{op} of G^{op} such that the map

$$G \longrightarrow G^{\mathrm{op}}, \quad g \longmapsto g^{\mathrm{op}}$$

is bijective, and the group structure on G^{op} is given by

$$g^{\mathrm{op}} \cdot h^{\mathrm{op}} = (h \cdot g)^{\mathrm{op}} \qquad \text{for all } g, h, \in G.$$

Every group G is isomorphic to its opposite group G^{op} via the inversion map

$$i \colon G \longrightarrow G^{\mathrm{op}}, \quad g \longmapsto (g^{\mathrm{op}})^{-1}.$$

Indeed, the map i bijective because it is the composite of the two bijections

$$G \longrightarrow G^{\mathrm{op}}, \quad g \longmapsto g^{\mathrm{op}}$$

and

$$G^{\mathrm{op}} \longrightarrow G^{\mathrm{op}}, \quad x \longmapsto x^{-1}.$$

It is a homomorphism of groups because

$$i(gh) = ((gh)^{\mathrm{op}})^{-1} = (h^{\mathrm{op}} g^{\mathrm{op}})^{-1} = (g^{\mathrm{op}})^{-1} (h^{\mathrm{op}})^{-1} = i(g) i(h)$$

for all $g, h \in G$.

(b)

We follow the approach from [MSE13b]. Let X be a set that contains at least two elements and let M be the monoid of maps from X to itself, i.e., let M be $\mathrm{End}_{\mathsf{Set}}(X)$.

Every constant map c from X to X is left-absorbing in M, in the sense that $cf = c$ for every element f of M. It follows that the opposite monoid M^{op} contains right-absorbing elements.

But M itself does not admit any right-absorbing element. Indeed, there exist for every element f of M two elements x and y of X with $f(x) \neq y$ (because the set X contains two distinct elements). Let s be the transposition on the set X that interchanges the two elements $f(x)$ and y. Then $(sf)(x) = y \neq f(x)$ and therefore $sf \neq f$.

Exercise 1.2.24

Suppose that such a functor Z were to exist. The symmetric group S_3 contains a transposition, which corresponds to a non-trivial homomorphism of groups f from $\mathbb{Z}/2$ to S_3. There also exists a non-trivial homomorphism of groups g from S_3 to $\mathbb{Z}/2$, which is given by

$$g(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is even,} \\ 1 & \text{if } \sigma \text{ is odd,} \end{cases}$$

for every permutation σ in S_3. The composite $g \circ f$ is the identity on $\mathbb{Z}/2$, whence we have the following commutative diagram:

$$\begin{tikzcd} & \mathrm{S}_3 \arrow[dr, "g"] & \\ \mathbb{Z}/2 \arrow[ur, "f"] \arrow[rr, "1_{\mathbb{Z}/2}"] & & \mathbb{Z}/2 \end{tikzcd}$$

By applying the functor Z to this diagram, we arrive at the following commutative diagram:

$$\begin{tikzcd} & \mathrm{Z}(\mathrm{S}_3) \arrow[dr, "\mathrm{Z}(g)"] & \\ \mathrm{Z}(\mathbb{Z}/2) \arrow[ur, "\mathrm{Z}(f)"] \arrow[rr, "1_{\mathrm{Z}(\mathbb{Z}/2)}"] & & \mathrm{Z}(\mathbb{Z}/2) \end{tikzcd}$$

The group $\mathbb{Z}/2$ is abelian and the center of the symmetric group S_3 it trivial. The above commutative diagram can therefore be rewritten as follows:

$$\begin{tikzcd} & 1 \arrow[dr] & \\ \mathbb{Z}/2 \arrow[ur] \arrow[rr, "1_{\mathbb{Z}/2}"] & & \mathbb{Z}/2 \end{tikzcd}$$

It follows from the commutativity of this last diagram that the identity homomorphism of $\mathbb{Z}/2$ is trivial. But this is false because the group $\mathbb{Z}/2$ is non-trivial.

Exercise 1.2.25

(a)

We first check that for every object A of $\mathscr{A}$, the assignment F^A is indeed a functor from $\mathscr{B}$ to $\mathscr{C}$.

- If g is a morphism from B to B' in $\mathscr{B}$ then the pair $(1_A, g)$ is a morphism from (A, B) to (A, B') in $\mathscr{A} \times \mathscr{B}$. It follows that $F(1_A, g)$ is a morphism from $F(A, B) = F^A(B)$ to $F(A, B') = F^A(B')$ in $\mathscr{C}$.
- We have for every object B of $\mathscr{B}$ the equalities

$$F^A(1_B) = F(1_A, 1_B) = F(1_{(A,B)}) = 1_{F(A,B)} = 1_{F^A(B)} .$$

- Let

$$g : B \longrightarrow B' , \quad g' : B' \longrightarrow B''$$

be two composable morphisms in $\mathscr{B}$. We have

$$\begin{aligned} F^A(g') \circ F^A(g) &= F(1_A, g') \circ F(1_A, g) \\ &= F((1_A, g') \circ (1_A, g)) \\ &= F(1_A \circ 1_A, g' \circ g) \\ &= F(1_A, g' \circ g) \\ &= F^A(g' \circ g) . \end{aligned}$$

This shows that the assignment F^A is indeed a functor from $\mathscr{B}$ to $\mathscr{C}$.

Let now B be a fixed object of the category $\mathscr{B}$. For every object A of $\mathscr{A}$ let $F_B(A)$ be the object $F(A, B)$ of $\mathscr{A} \times \mathscr{B}$. For every morphism

$$f : A \longrightarrow A'$$

in $\mathscr{A}$ let $F_B(f)$ be the morphism $F(f, 1_B)$ in $\mathscr{C}$. The domain of this morphism is $(A, B) = F_B(A)$, and its codomain is $(A', B) = F_{B'}(A)$.

- We have for every object A of $\mathscr{A}$ the chain of equalities

$$F_B(1_A) = F(1_A, 1_B) = F(1_{(A,B)}) = 1_{F(A,B)} = 1_{F_B(A)} .$$

- For every two composable morphisms

$$f : A \longrightarrow A' , \quad f' : A' \longrightarrow A''$$

  in $\mathscr{A}$, we have the chain of equalities

$$\begin{aligned} F_B(f') \circ F_B(f) &= F(f', 1_B) \circ F(f, 1_B) \\ &= F((f', 1_B) \circ (f, 1_B)) \\ &= F(f' \circ f, 1_B \circ 1_B) \\ &= F(f' \circ f, 1_B) \\ &= F_B(f' \circ f) . \end{aligned}$$

This shows that F_B is a functor from $\mathscr{A}$ to $\mathscr{C}$.

(b)

We have for every object A of $\mathscr{A}$ and every object B of $\mathscr{B}$ the equalities

$$F^A(B) = (A, B) = F_B(A) .$$

We also have for all morphisms

$$f : A \longrightarrow A' , \quad g : B \longrightarrow B'$$

in $\mathscr{A}$ and $\mathscr{B}$ respectively the chains of equalities

$$F^{A'}(g) \circ F_B(f) = F(1_{A'}, g) \circ F(f, 1_B) = F((1_{A'}, g) \circ (f, 1_B)) = F(f, g)$$

and

$$F_{B'}(f) \circ F^A(g) = F(f, 1_{B'}) \circ F(1_A, g) = F((f, 1_{B'}) \circ (1_A, g)) = F(f, g) ,$$

and therefore the equalities

$$F^{A'}(g) \circ F_B(f) = F(f, g) = F_{B'}(f) \circ F^A(g) .$$

(c)

Suppose first that such a functor F were to exist. Then

$$F(A,B) = F^A(B)$$

for every object (A,B) of $\mathscr{A} \times \mathscr{B}$, and

$$F(f,g) = F((1_{A'},g) \circ (f,1_B)) = F(1_{A'},g) \circ F(f,1_B) = F^{A'}(g) \circ F_B(f)$$

for every morphism (f,g) from (A,B) to (A',B') in $\mathscr{A} \times \mathscr{B}$. This shows the uniqueness of F. In the following, we will show the existence of F.

For every object A of $\mathscr{A}$ and B of $\mathscr{B}$, we denote the element $F^A(B)$ of $\mathscr{C}$ by $F(A,B)$. We have equivalently $F(A,B) = F_B(A)$. For every morphism

$$(f,g) \colon (A,B) \longrightarrow (A',B')$$

in $\mathscr{A} \times \mathscr{B}$ we define

$$F(f,g) \coloneqq F^{A'}(g) \circ F_B(f)\,.$$

Equivalently,

$$F(f,g) = F_{B'}(f) \circ F^A(g)\,.$$

We have to check that F is indeed a functor.

- Let

  $$(f,g) \colon (A,B) \longrightarrow (A',B')$$

  be a morphism in $\mathscr{A} \times \mathscr{B}$. The morphism $F(f,g) = F^{A'}(g) \circ F_B(f)$ goes from $F(A,B)$ to $F(A',B')$, as we can from the following diagram:

$$\begin{tikzcd}
F_B(A) \arrow[r, "F_B(f)"] \arrow[d, equal] & F_B(A') \arrow[r, equal] \arrow[d, equal] & F^{A'}(B) \arrow[r, "F^{A'}(g)"] \arrow[d, equal] & F^{A'}(B') \arrow[d, equal] \\
F(A,B) \arrow[r] & F(A',B) \arrow[r, equal] & F(A',B) \arrow[r] & F(A',B')
\end{tikzcd}$$

- We have for every object (A,B) of $\mathscr{A} \times \mathscr{B}$ the chain of equalities

$$\begin{aligned} F(1_{(A,B)}) = F(1_A,1_B) = F^A(1_B) \circ F_B(1_A) = 1_{F^A(B)} \circ 1_{F_B(A)} &= 1_{F(A,B)} \circ 1_{F(A,B)} \\ &= 1_{F(A,B)}\,. \end{aligned}$$

- For every two composable morphisms
$$(f,g)\colon\ (A,B) \longrightarrow (A',B')\,,\quad (f',g')\colon\ (A',B') \longrightarrow (A'',B'')$$
in $\mathscr{A}\times\mathscr{B}$, we have the chain of equalities
$$\begin{aligned} F(f',g')\circ F(f,g) &= F^{A''}(g')\circ F_{B'}(f')\circ F_{B'}(f)\circ F^{A}(g) \\ &= F^{A''}(g')\circ F_{B'}(f'\circ f)\circ F^{A}(g) \\ &= F^{A''}(g')\circ F^{A''}(g)\circ F_{B}(f'\circ f) \\ &= F^{A''}(g'\circ g)\circ F_{B}(f'\circ f) \\ &= F(g'\circ g, f'\circ f) \\ &= F((g',f')\circ(g,f)) \\ &= F(g',f')\circ F(g,f)\,. \end{aligned}$$

This shows altogether that the assignment F is indeed a functor.

Exercise 1.2.26

For every topological space X let $C(X)$ be the (commutative) ring of continuous, real-valued functions on X. For any two topological spaces X and Y and every continuous map f from X to Y, let $C(f)$ be the induced map
$$C(f)\colon\ C(Y) \longrightarrow C(X)\,,\quad \varphi \longmapsto \varphi\circ f\,.$$
This map is well-defined because the composite of two continuous maps is again continuous.

Let us check that map $C(f)$ is a homomorphism of rings:

- If we denote by $\star$ either addition or multiplication, then we have for every two elements φ and ψ of $C(Y)$ the equalities
$$\begin{aligned} C(f)(\varphi\star\psi)(x) &= ((\varphi\star\psi)\circ f)(x) \\ &= (\varphi\star\psi)(f(x)) \\ &= \varphi(f(x))\star\psi(\varphi(x)) \\ &= (\varphi\circ f)(x)\star(\psi\circ\varphi)(x) \\ &= C(f)(\varphi)(x)\star C(\varphi)(\psi)(x) \\ &= (C(f)(\varphi)\star C(\varphi)(\psi))(x) \end{aligned}$$
for every point x in X, and thus the equality
$$C(f)(\varphi\star\psi) = C(f)(\varphi)\star C(f)(\psi)\,.$$

- Let us denote by 1_X and 1_Y the functions with constant value 1 on X and Y respectively. We have

$$C(f)(1_Y) = 1_Y \circ f = 1_X\,,$$

and thus $C(f)(1) = 1$.

These calculations show that the map $C(f)$ is indeed a homomorphism of rings from $C(Y)$ to $C(X)$.

Let us now check the functoriality of the construction C.

- For every topological space X we have

$$C(1_X)(\varphi) = \varphi \circ 1_X = \varphi = 1_{C(X)}(\varphi)$$

for every $\varphi \in C(X)$, and therefore $C(1_X) = 1_{C(X)}$. (Here we denote by 1_X the identity map on the space X.)

- For every two composable continuous maps

$$f : X \longrightarrow Y\,, \quad g : Y \longrightarrow Z\,,$$

we have the chain of equalities

$$\begin{aligned} C(g \circ f)(\varphi) &= \varphi \circ g \circ f \\ &= C(f)(\varphi \circ g) \\ &= C(f)(C(g)(\varphi)) \\ &= (C(f) \circ C(g))(\varphi) \end{aligned}$$

for every $\varphi \in C(Z)$, and therefore $C(g \circ f) = C(g) \circ C(f)$.

This shows that C is indeed a contravariant functor from **Top** to **Ring**.

Exercise 1.2.27

Let **2** be the category with two objects, named X and Y, and one non-identity morphism, named f, which goes from X to Y. This category looks as follows:

$$X \xrightarrow{\quad f \quad} Y$$

Let $\mathbf{2}+\mathbf{2}$ be the category that consists of two copies of $\mathbf{2}$. More precisely, the category $\mathbf{2}+\mathbf{2}$ looks as follows:

$$
\begin{tikzcd}
X_1 \arrow[r, "f_1"] & Y_1 \\
X_2 \arrow[r, "f_2"] & Y_2
\end{tikzcd}
$$

Let F be the functor from $\mathbf{2}+\mathbf{2}$ to $\mathbf{2}$ that maps the two copies of $\mathbf{2}$ in $\mathbf{2}+\mathbf{2}$ back onto $\mathbf{2}$. More explicitly,

$$
F(X_1), F(X_2) = X\,, \quad F(Y_1), F(Y_2) = Y\,, \quad F(f_1), F(f_2) = f\,.
$$

The functor F may be depicted as follows:

$$
\begin{tikzcd}
X_1 \arrow[r, "f_1"] & Y_1 & & \\
 & & \rightsquigarrow & X \arrow[r, "f"] & Y \\
X_2 \arrow[r, "f_2"] & Y_2 & &
\end{tikzcd}
$$

The functor F is faithful because for every two objects A and A' of $\mathbf{2}+\mathbf{2}$, the set $(\mathbf{2}+\mathbf{2})(A, A')$ contains at most one element. But f_1 and f_2 are two distinct morphisms in $\mathbf{2}+\mathbf{2}$ that have the same image under F.

Exercise 1.2.28

(a)

Example 1.2.3, (a). The forgetful functor U from **Grp** to **Set** is faithful. But it is not full: Let G be the trivial group, let H be a non-trivial group and let h be a non-identity element of H. The map $U(G) \to U(H)$ given by $1 \mapsto h$ is not a homomorphism of groups, and is therefore not contained in the image of U.

Example 1.2.3, (b). The forgetful functor U from **Ring** to **Set** is faithful. But it is not full: Let both R and S be the ring of integers, i.e., the ring $\mathbb{Z}$. The map

$$
f\colon \mathbb{Z} \longrightarrow \mathbb{Z}\,, \quad n \longmapsto 2n
$$

is not a homomorphism of rings. It is therefore not contained in the image of U.

Example 1.2.3, (c). The forgetful functor from **Ring** to **Ab** is faithful. But it is not full: this can be seen from the same counterexample as for the previous functor.

The forgetful functor U from **Ring** to **Mon** is faithful. But it is not full: Let both R and S be the ring of integers, i.e., the ring $\mathbb{Z}$, and let f be the trivial homomorphism of monoids from $U(R)$ to $U(S)$. The map f is not a homomorphism of rings from R to S, and therefore not contained in the image of U.

Example 1.2.3, (d). The inclusion functor from **Ab** to **Grp** is both faithful and full since **Ab** is a full subcategory of **Grp**.

Example 1.2.4, (a). The free functor F from **Set** to **Grp** is faithful: Let S and T be two sets and let f be a map from S to T. The induced homomorphism of groups $F(f)$ from $F(S)$ to $F(T)$ satisfies the condition

$$F(f)(s) = f(s)$$

for every element s of S. The original map f is therefore uniquely determined by its induced homomorphism of groups $F(f)$.

But the functor F is not full: Let S be a non-empty set and let T be the empty set. There exists no map from S to T, but there exist a homomorphism of groups from $F(S)$ to $F(T)$ (namely the trivial homomorphism).

Example 1.2.4, (b). The free functor from **Set** to **CRing** is faithful, but not full. This can be seen as in the previous example.

Example 1.2.4, (c). The free functor from **Set** to $\mathbf{Vect}_{\mathbb{k}}$ is faithful, but not full. This can be seen as in the previous example.

Example 1.2.5, (a). The functor π_1 from $\mathbf{Top}_*$ to **Grp** is not faithful. To see this, we consider the pointed topological space $(\mathbb{R}, 0)$. The fundamental group $\pi_1(\mathbb{R}, 0)$ is trivial, whence there exists precisely one homomorphism of groups from $\pi_1(\mathbb{R}, 0)$ to $\pi_1(\mathbb{R}, 0)$. But there exist many more continuous maps from $(\mathbb{R}, 0)$ to $(\mathbb{R}, 0)$.[2]

The functor π_1 is also not full. The author doesn't know a good example for this, and refers to [MSE13c].

Example 1.2.5, (b). The author strongly suspects that these functors are neither faithful nor full, but isn't good enough at topology to give examples that confirm these suspicions.

[2] Namely, $2^{\aleph_0}$ many.

Example 1.2.7. The functor F from $\mathscr{G}$ to $\mathscr{H}$ is faithful if and only if the corresponding homomorphism of groups f from G to H is injective. Similarly, the functor F is full if and only if the corresponding homomorphism f is surjective.

Example 1.2.8. This functor is faithful if and only if the corresponding G-set, respectively representation of G, is faithful.

Example 1.2.9. This functor is faithful, because there exists for any two elements a and a' of A at most one morphism from a to a' in $\mathscr{A}$. It is full if and only if it follows for any two elements a and a' of $\mathscr{A}$ from $f(a) \leq f(a')$ that also $a \leq a'$.

Example 1.2.11. The given functor C from **Top** to **Ring** is not faithful. To see this, let $X := \{x\}$ be the one-point topological space and let $Y := \{y_1, y_2\}$ be the two-point indiscrete topology space. Both $C(X)$ and $C(Y)$ consist only of constant functions, and the two continuous maps from X to Y induce the same homomorphism of rings from $C(Y)$ to $C(X)$.

The functor C is also not full. The author doesn't have a (counter)example for this, and refers to [MSE17].

Example 1.2.12. The functor $(-)^*$ from $\mathbf{Vect}_{\Bbbk}$ to $\mathbf{Vect}_{\Bbbk}$ is faithful. To see this, let V and W be two $\Bbbk$-vector spaces and let f_1 and f_2 be two distinct linear maps from V to W. There exists by assumption a vector v in V for which the two vectors $w_1 := f_1(v)$ and $w_2 := f_2(v)$ in W are distinct. It follows that there exists some linear functional w^* in W^* with $w^*(w_1) \neq w^*(w_2)$. It further follows that

$$
\begin{aligned}
f_1^*(w^*)(v) &= (w^* \circ f_1)(v) \\
&= w^*(f_1(w)) \\
&= w^*(w_1) \\
&\neq w^*(w_2) \\
&= \cdots \\
&= f_2^*(w^*)(v)\,.
\end{aligned}
$$

This shows that $f_1^*(w^*) \neq f_2^*(w^*)$, and therefore $f_1^* \neq f_2^*$.

However, the functor $(-)^*$ is not full. To see this, we observe that for any two $\Bbbk$-vector spaces V and W, the map

$$
D_{V,W} \colon \; \mathrm{Hom}_{\Bbbk}(V, W) \longrightarrow \mathrm{Hom}_{\Bbbk}(W^*, V^*)\,, \quad f \longmapsto f^*
$$

is linear. For $V = \Bbbk$, the domain of $D_{V,W}$ is given by

$$\mathrm{Hom}_{\Bbbk}(V, W) = \mathrm{Hom}_{\Bbbk}(\Bbbk, W) \cong W\,,$$

while its codomain is given by

$$\mathrm{Hom}_{\Bbbk}(W^*, V^*) = \mathrm{Hom}_{\Bbbk}(W^*, \Bbbk^*) \cong \mathrm{Hom}_{\Bbbk}(W^*, \Bbbk) = W^{**}\,.$$

If we choose the vector space W to be infinite-dimensional, then it follows that the map $D_{\Bbbk,W}$ cannot be surjective, since W^{**} is of strictly larger dimension than W.

(b)

Let $\mathbf{2}$ be the category given by two objects 0 and 1 and one non-identity morphism i, going from 0 to 1. This category may be depicted as follows:

$$\mathbf{2}: \quad 0 \xrightarrow{\;\;i\;\;} 1$$

Let similarly $\mathbf{2}'$ be the category given by two objects 0 and 1 and two parallel non-identity morphisms from 0 to 1, denoted by i and i'. This category may be depicted as follows:

$$\mathbf{2}': \quad 0 \underset{i'}{\overset{i}{\rightrightarrows}} 1$$

The inclusion functor

$$I : \mathbf{2} \longrightarrow \mathbf{2}' \quad \text{with} \quad F(0) = 0\,, \quad F(1) = 1\,, \quad F(i) = i$$

is faithful but not full. The retraction functor

$$R : \mathbf{2}' \longrightarrow \mathbf{2} \quad \text{with} \quad F(0) = 0\,, \quad F(1) = 1\,, \quad F(i) = i \quad F(i') = i$$

is full but not faithful. The composite $R \circ I$ is the identity functor on $\mathbf{2}$, which is both full and faithful. The composite $I \circ R$ is neither full nor faithful.

Exercise 1.2.29

(a)

Let $(P, \leq)$ be an ordered set and let $\mathscr{P}$ be the corresponding category. A subcategory $\mathscr{Q}$ of $\mathscr{P}$ is the same as a subset Q of P together with a partial order $\preceq$

on Q such that $x \leq y$ whenever $x \preceq y$. In other words, a subcategory $\mathscr{Q}$ of $\mathscr{P}$ is the same as an ordered set Q that is subordinate to P. Such a subcategory is full if and only if it follows conversely for any two elements x and y of Q from $x \leq y$ that also $x \preceq y$.

(b)

Let G be a group and let $\mathscr{G}$ be the corresponding one-object category. A subcategory $\mathscr{H}$ of $\mathscr{G}$ is the same as a submonoid H of G or the empty set. The only full subcategories of $\mathscr{G}$ are G itself and the empty set.

1.3 Natural transformations

Exercise 1.3.25

- We have the two functors

$$T_1, T_2 \colon \mathbf{Vect}_{\Bbbk} \times \mathbf{Vect}_{\Bbbk} \longrightarrow \mathbf{Vect}_{\Bbbk}$$

  given by

$$\begin{aligned} T_1(V,W) &= V \otimes W\,, & T_1(f,g) &= f \otimes g\,, \\ T_2(V,W) &= W \otimes V\,, & T_2(f,g) &= g \otimes f\,. \end{aligned}$$

  We have for every two $\Bbbk$-vector spaces V and W the isomorphism of vector spaces

$$\sigma_{(V,W)} \colon V \otimes W \longrightarrow W \otimes V\,, \quad v \otimes w \longmapsto w \otimes v\,.$$

  This isomorphism is natural in (V,W), in the sense that these isomorphisms assemble into a natural isomorphism σ from T_1 to T_2.

- We have a functor

$$D \colon \mathbf{Grp} \longrightarrow \mathbf{Grp}\,, \quad G \longmapsto G^{\mathrm{op}}\,, \quad f \longmapsto f\,.$$

  We have for every group G an isomorphism of groups given by

$$\alpha_G \colon G \longrightarrow G^{\mathrm{op}}\,, \quad g \longmapsto g^{\mathrm{op}}\,.$$

  These isomorphisms are natural in G, in the sense that they assemble in a natural isomorphism α from $1_{\mathbf{Grp}}$ to D.

- We have the two contravariant functors

  $$D_1, D_2 \colon \mathbf{Vect}_{\Bbbk} \times \mathbf{Vect}_{\Bbbk} \longrightarrow \mathbf{Vect}_{\Bbbk}$$

  given by

  $$T_1(V, W) = V^* \otimes W^*\,, \quad T_1(f, g) = f^* \otimes g^*\,,$$

  and

  $$T_2(V, W) = (V \otimes W)^*\,, \quad T_2(f, g) = (f \otimes g)^*\,.$$

  We have for every two $\Bbbk$-vector spaces V and W a linear map

  $$\begin{aligned} \beta_{(V,W)} \colon V^* \otimes W^* &\longrightarrow (V \otimes W)^*\,, \\ v^* \otimes w^* &\longmapsto [v \otimes w \longmapsto v^*(v) \cdot w^*(w)]\,. \end{aligned}$$

  This linear map is natural in (V, W), in the sense that these linear maps induce a natural transformation β from T_1 to T_2.

Exercise 1.3.26

Suppose first that α is a natural isomorphism. This means that there exists a natural transformation β from G to F such that both $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$. For every object A of $\mathscr{A}$, we have therefore the equality

$$\beta_A \circ \alpha_A = (\beta \circ \alpha)_A = (1_F)_A = 1_{F(A)}$$

and similarly the equality

$$\alpha_A \circ \beta_A = (\alpha \circ \beta)_A = (1_G)_A = 1_{G(A)}\,.$$

This shows that the morphism α_A is an isomorphism whose inverse is given by β_A.

Suppose now that for every object A of $\mathscr{A}$ the morphism α_A is an isomorphism, and let β_A be the inverse of α_A, i.e., $\beta_A \coloneqq \alpha_A^{-1}$. We have for every morphism

$$f \colon A \longrightarrow A'$$

in $\mathscr{A}$ the commutative square diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"] & G(A') \end{tikzcd}$$

by the naturality of α. The commutativity of this diagram is equivalent to the equality

$$\alpha_{A'} \circ F(f) = G(f) \circ \alpha_A \,,$$

which in turn is equivalent to the equality

$$F(f) \circ \alpha_A^{-1} = \alpha_{A'}^{-1} \circ G(f) \,.$$

In other words,

$$F(f) \circ \beta_A = \beta_{A'} \circ G(f) \,,$$

which tells us that the following square diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] & F(A') \\ G(A) \arrow[u, "\beta_A"] \arrow[r, "G(f)"] & G(A') \arrow[u, "\beta_{A'}"'] \end{tikzcd}$$

This shows that $\beta = (\beta_A)_{A \in \mathrm{Ob}(\mathscr{A})}$ is a natural transformation from G to F.

We have for every object A of $\mathscr{A}$ the equalities

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A = \alpha_A^{-1} \circ \alpha_A = 1_{F(A)} = (1_F)_A \,,$$

which shows that $\beta \circ \alpha = 1_F$. We find in the same way that also $\alpha \circ \beta = 1_G$. This shows that α is a natural isomorphism whose inverse is given by β.[3]

Exercise 1.3.27

Let F be a functor from $\mathscr{A}$ to $\mathscr{B}$. We can regard F as a functor from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}^{\mathrm{op}}$, which we shall denote by F' instead of F for clarity. This functor F' is given by

$$F'(A^{\mathrm{op}}) = F(A)^{\mathrm{op}} \quad \text{and} \quad F'(f^{\mathrm{op}}) = F(f)^{\mathrm{op}}$$

for every object A of $\mathscr{A}$ and every morphism f in $\mathscr{A}$. This assignment F' is indeed a functor from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}^{\mathrm{op}}$:

- We have for every object A of $\mathscr{A}$ the equalities

$$F'(1_{A^{\mathrm{op}}}) = F'(1_A^{\mathrm{op}}) = F(1_A)^{\mathrm{op}} = 1_{F(A)}^{\mathrm{op}} = 1_{F(A)^{\mathrm{op}}} \,.$$

[3] In other words, $(\alpha^{-1})_A = (\alpha_A)^{-1}$ for every object A of $\mathscr{A}$. This allows us to just write α_A^{-1} for this morphism.

- We have for every two composable morphisms

$$f : A \longrightarrow A', \quad g : A' \longrightarrow A''$$

  in $\mathscr{A}$ the equalities

$$\begin{aligned} F'(f^{\mathrm{op}} g^{\mathrm{op}}) &= F'((gf)^{\mathrm{op}}) \\ &= F(gf)^{\mathrm{op}} \\ &= (F(g)F(f))^{\mathrm{op}} \\ &= F(f)^{\mathrm{op}} F(g)^{\mathrm{op}} \\ &= F'(f^{\mathrm{op}}) F'(g^{\mathrm{op}}) . \end{aligned}$$

This shows that F' is indeed a functor from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}^{\mathrm{op}}$. We have therefore the map

$$(-)' : \mathrm{Ob}([\mathscr{A}, \mathscr{B}]) \longrightarrow \mathrm{Ob}([\mathscr{A}^{\mathrm{op}}, \mathscr{B}^{\mathrm{op}}]) . \tag{1.1}$$

Let F and G be two functors from $\mathscr{A}$ to $\mathscr{B}$ and let α be a natural transformation from F to G. We have for every morphism

$$f : A \longrightarrow A'$$

in $\mathscr{A}$ the following commutative square diagram in $\mathscr{B}$:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"] & G(A') \end{tikzcd}$$

It follows that we have the following commutative diagram in $\mathscr{B}^{\mathrm{op}}$:

$$\begin{tikzcd} F(A)^{\mathrm{op}} & F(A')^{\mathrm{op}} \arrow[l, "F(f)^{\mathrm{op}}"'] \\ G(A)^{\mathrm{op}} \arrow[u, "\alpha_A^{\mathrm{op}}"] & G(A')^{\mathrm{op}} \arrow[l, "G(f)^{\mathrm{op}}"'] \arrow[u, "\alpha_{A'}^{\mathrm{op}}"'] \end{tikzcd}$$

This diagram can also be written as follows:

$$\begin{tikzcd} F'(A^{\mathrm{op}}) & F'(A'^{\mathrm{op}}) \arrow[l, "F'(f^{\mathrm{op}})"'] \\ G'(A^{\mathrm{op}}) \arrow[u, "\alpha_A^{\mathrm{op}}"] & G'(A'^{\mathrm{op}}) \arrow[l, "G'(f^{\mathrm{op}})"'] \arrow[u, "\alpha_{A'}^{\mathrm{op}}"'] \end{tikzcd}$$

The commutativity of the above diagram shows we have a natural transformation α' from G' to F' with components $(\alpha')_{A^{\mathrm{op}}} \coloneqq \alpha_A^{\mathrm{op}}$ for every object A of $\mathscr{A}$.

The natural transformation α' depends contravariantly functorial on α:

- Let F be a functor from $\mathscr{A}$ from $\mathscr{B}$. We have the chain of equalities

$$(1'_F)_{A^{\mathrm{op}}} = 1_F(A)^{\mathrm{op}} = 1_{F(A)}^{\mathrm{op}} = 1_{F(A)^{\mathrm{op}}} = 1_{F'(A^{\mathrm{op}})} = 1_{F'}(A^{\mathrm{op}})$$

  for every object A of $\mathscr{A}$. This shows that $1'_F = 1_{F'}$.
- Let F, G and H be functors from $\mathscr{A}$ to $\mathscr{B}$, and let

$$\alpha : F \Longrightarrow G \,, \quad \beta : G \Longrightarrow H$$

  be natural transformations. We have the induces natural transformation

$$\alpha' : G' \Longrightarrow F' \,, \quad \beta' : H' \Longrightarrow G' \,, \quad (\beta\alpha)' : H' \Longrightarrow F' \,.$$

  We have for every object A of $\mathscr{A}$ the chain of equalities

$$(\alpha'\beta')_{A^{\mathrm{op}}} = (\alpha')_{A^{\mathrm{op}}}(\beta')_{A^{\mathrm{op}}} = \alpha_A^{\mathrm{op}}\beta_A^{\mathrm{op}} = (\beta_A\alpha_A)^{\mathrm{op}} = (\beta\alpha)_A^{\mathrm{op}} = ((\beta\alpha)')_{A^{\mathrm{op}}} \,,$$

  and therefore altogether the equality

$$\alpha'\beta' = (\beta\alpha)' \,.$$

We have altogether shown that the mapping (1.1) extends to a contravariant functor

$$D_{\mathscr{A},\mathscr{B}} : [\mathscr{A},\mathscr{B}] \longrightarrow [\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}] \,,$$

We claim that the two contravariant functors $D_{\mathscr{A},\mathscr{B}}$ and $D_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}}$ are mutually inverse. This then shows that $D_{\mathscr{A},\mathscr{B}}$ is an isomorphism of categories from $[\mathscr{A},\mathscr{B}]^{\mathrm{op}}$ to $[\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}]$.

Indeed, we have

$$D_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}}(D_{\mathscr{A},\mathscr{B}}(F)) = D_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}}(F') = F'' = F$$

for every object F of $[\mathscr{A},\mathscr{B}]$, and similarly

$$D_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}}(D_{\mathscr{A},\mathscr{B}}(\alpha)) = D_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}}(\alpha') = \alpha'' = \alpha$$

for every morphism α in $[\mathscr{A},\mathscr{B}]$. These equalities show that the composite $D_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}} \circ D_{\mathscr{A},\mathscr{B}}$ is the identity functor of $[\mathscr{A},\mathscr{B}]$. It follows that also

$$D_{\mathscr{A},\mathscr{B}} \circ D_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}} = D_{\mathscr{A}^{\mathrm{opop}},\mathscr{B}^{\mathrm{opop}}} \circ D_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}} = 1_{[\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}]} \,.$$

Exercise 1.3.28

(a)

For any two sets A and B, we can consider the evaluation map

$$\alpha_{A,B} \colon A \times B^A \longrightarrow B\,, \quad (a, f) \longmapsto f(a)\,.$$

(b)

For any two sets A and B, we consider the map

$$\beta_{A,B} \colon A \longrightarrow B^{(B^A)}\,, \quad a \longmapsto [f \longmapsto f(a)]\,.$$

Exercise 1.3.29

Suppose first that α is a natural transformation from F to G.

- Let B be any object of $\mathscr{B}$. We have for every morphism

  $$f \colon A \longrightarrow A'$$

  in $\mathscr{A}$ the following commutative square diagram:

  $$\begin{tikzcd} F(A,B) \arrow[r, "F(f, 1_B)"] \arrow[d, "\alpha_{A,B}"'] & F(A',B) \arrow[d, "\alpha_{A',B}"] \\ G(A,B) \arrow[r, "G(f, 1_B)"] & G(A',B) \end{tikzcd}$$

  This diagram can equivalently be written as follows:

  $$\begin{tikzcd} F_B(A) \arrow[r, "F_B(f)"] \arrow[d, "\alpha_{A,B}"'] & F_B(A') \arrow[d, "\alpha_{A',B}"] \\ G_B(A) \arrow[r, "G_B(f)"] & G_B(A') \end{tikzcd}$$

  The commutativity of this diagram tells us that the family $(\alpha_{A,B})_{A \in \mathrm{Ob}(\mathscr{A})}$ is a natural transformation from F_B to G_B.

- Let A be an object of $\mathscr{A}$. We have for every morphism

  $$g : B \longrightarrow B'$$

  in $\mathscr{B}$ the following commutative square diagram:

  $$\begin{tikzcd} F(A,B) \arrow[r, "F(1_A{,}g)"] \arrow[d, "\alpha_{A,B}"'] & F(A,B') \arrow[d, "\alpha_{A,B'}"] \\ G(A,B) \arrow[r, "G(1_A{,}g)"] & G(A,B') \end{tikzcd}$$

  This diagram can equivalently be written as follows:

  $$\begin{tikzcd} F^A(B) \arrow[r, "F^A(g)"] \arrow[d, "\alpha_{A,B}"'] & F^A(B') \arrow[d, "\alpha_{A,B'}"] \\ G^A(B) \arrow[r, "G^A(g)"] & G^A(B') \end{tikzcd}$$

  The commutativity of this diagram shows that the family $(\alpha_{A,B})_{B\in\mathrm{Ob}(\mathscr{B})}$ is a natural transformation from F^A to G^A.

Suppose now conversely that the following two assertions hold: for every object A of $\mathscr{A}$, the family $(\alpha_{A,B})_{B\in\mathrm{Ob}(\mathscr{B})}$ is a natural transformation from F^A to G^A, and for every object B of $\mathscr{B}$, the family $(\alpha_{A,B})_{A\in\mathrm{Ob}(\mathscr{A})}$ is a natural transformation from F_B to G_B. We need to show that the family $(\alpha_{A,B})_{(A,B)\in\mathrm{Ob}(\mathscr{A}\times\mathscr{B})}$ is a natural transformation from F to G. Let

$$(f,g) : (A,B) \longrightarrow (A',B')$$

be a morphism in $\mathscr{A}\times\mathscr{B}$. We have by assumption the following two commutative square diagrams:

$$\begin{tikzcd} F_B(A) \arrow[r, "F_B(f)"] \arrow[d, "\alpha_{A,B}"'] & F_B(A') \arrow[d, "\alpha_{A',B}"] \\ G_B(A) \arrow[r, "G_B(f)"] & G_B(A') \end{tikzcd} \qquad \begin{tikzcd} F^{A'}(B) \arrow[r, "F^{A'}(g)"] \arrow[d, "\alpha_{A',B}"'] & F^{A'}(B') \arrow[d, "\alpha_{A',B'}"] \\ G^{A'}(B) \arrow[r, "G^{A'}(g)"] & G^{A'}(B') \end{tikzcd}$$

We can equivalently write these diagrams as follows:

$$\begin{tikzcd} F(A,B) \arrow[r, "{F(f,1_B)}"] \arrow[d, "\alpha_{A,B}"'] & F(A',B) \arrow[d, "\alpha_{A',B}"] \\ G(A,B) \arrow[r, "{G(f,1_B)}"] & G(A',B) \end{tikzcd} \qquad \begin{tikzcd} F(A',B) \arrow[r, "{F(1_{A'},g)}"] \arrow[d, "\alpha_{A',B}"'] & F(A',B') \arrow[d, "\alpha_{A',B'}"] \\ G(A',B) \arrow[r, "{G(1_{A'},g)}"] & G(A',B') \end{tikzcd}$$

This rewriting allows us combine both diagrams into the following commutative diagram:

$$\begin{tikzcd} F(A,B) \arrow[r, "{F(f,1_B)}"] \arrow[d, "\alpha_{A,B}"'] & F(A',B) \arrow[r, "{F(1_{A'},g)}"] \arrow[d, "\alpha_{A',B}"] & F(A',B') \arrow[d, "\alpha_{A',B'}"] \\ G(A,B) \arrow[r, "{G(f,1_B)}"] & G(A',B) \arrow[r, "{G(1_{A'},g)}"] & G(A',B') \end{tikzcd}$$

By leaving out the middle column this combined diagram, we arrive at the following commutative diagram:

$$\begin{tikzcd} F(A,B) \arrow[r, "{F(1_{A'},g)\circ F(f,1_B)}"] \arrow[d, "\alpha_{A,B}"'] & F(A',B') \arrow[d, "\alpha_{A',B'}"] \\ G(A,B) \arrow[r, "{G(1_{A'},g)\circ G(f,1_B)}"] & G(A',B') \end{tikzcd}$$

Thanks to the equalities

$$F(1_{A'}, g) \circ F(f, 1_B) = F((1_{A'}, g) \circ (f, 1_B)) = F(f, g)$$

and

$$G(1_{A'}, g) \circ G(f, 1_B) = G((1_{A'}, g) \circ (f, 1_B)) = G(f, g)\,,$$

we can further rewrite this commutative diagram as follows:

$$\begin{tikzcd} F(A,B) \arrow[r, "{F(f,g)}"] \arrow[d, "\alpha_{A,B}"'] & F(A',B') \arrow[d, "\alpha_{A',B'}"] \\ G(A,B) \arrow[r, "{G(f,g)}"] & G(A',B') \end{tikzcd}$$

The commutativity of this final diagram shows that α is a natural transformation from F to G.

Exercise 1.3.30

The author suspects that the result will be conjugation. Let us check in the following that the author is correct.

Let $\mathscr{G}$ be the one-object category corresponding to G. Let g and h be two elements of the group G and let φ and ψ be the corresponding homomorphism of groups from $\mathbb{Z}$ to G, given by

$$\varphi(n) := g^n \quad \text{and} \quad \psi(n) := h^n$$

for every integer n. The corresponding functors Φ and Ψ are isomorphic if and only if there exists a natural transformation from Φ to Ψ: every morphism in $\mathscr{G}$ is already an isomorphism, and every natural transformation from Φ to Ψ is therefore already a natural isomorphism by Lemma 1.3.11.

There exists a natural transformation from Φ to Ψ if and only if there exists a morphism k in $\mathscr{G}$, i.e., an element of G, such that the following square diagram commutes for every integer n:

$$\begin{tikzcd} \Phi(*) \arrow[r, "\Phi(n)"] \arrow[d, "k"'] & \Phi(*) \arrow[d, "k"] \\ \Psi(*) \arrow[r, "\Psi(n)"] & \Psi(*) \end{tikzcd}$$

We hence need $k \circ \Phi(n) = \Psi(n) \circ k$, or equivalently $k \circ \varphi(n) = \psi(n) \circ k$, or equivalently

$$kg^n = h^n k$$

for every integer n. For $n = 1$ this gives us the necessary condition $kgk^{-1} = h$. This condition is also sufficient because the cases $n \neq 1$ follow from it.

We hence find that natural transformations from Φ to Ψ correspond bijectively to elements k of G for which $kgk^{-1} = h$. It follows that the two group elements g and h define isomorphic functors if and only if they are conjugated.

Exercise 1.3.31

(a)

Let X and Y be two finite sets and let f be a bijection from X to Y (i.e., a morphism in the given category $\mathscr{B}$). We let $\mathrm{Sym}(f)$ be the resulting conjugation isomorphism of groups from $\mathrm{Sym}(X)$ to $\mathrm{Sym}(Y)$, i.e., the map

$$\mathrm{Sym}(f)\colon\ \mathrm{Sym}(X) \longrightarrow \mathrm{Sym}(Y)\,, \quad s \longmapsto f \circ s \circ f^{-1}\,.$$

This construction defines a functor Sym from $\mathscr{B}$ to **Set**:

- For every set X we have

  $$\mathrm{Sym}(1_X)(s) = 1_X \circ s \circ 1_X^{-1} = 1_X \circ s \circ 1_X = s = 1_{\mathrm{Sym}(X)}(s)$$

  for every $s \in \mathrm{Sym}(X)$, and therefore $\mathrm{Sym}(1_X) = 1_{\mathrm{Sym}(X)}$.
- We have for any two composable morphisms

  $$f\colon\ X \longrightarrow Y, \quad g\colon\ Y \longrightarrow Z$$

  in $\mathscr{B}$, the equalities

  $$\begin{aligned} \mathrm{Sym}(g)(\mathrm{Sym}(f)(s)) &= g \circ f \circ s \circ f^{-1} \circ g^{-1} \\ &= (g \circ f) \circ s \circ (g \circ f)^{-1} \\ &= \mathrm{Sym}(g \circ f)(s) \end{aligned}$$

  for every $s \in \mathrm{Sym}(X)$, and therefore the equality

  $$\mathrm{Sym}(g \circ f) = \mathrm{Sym}(g) \circ \mathrm{Sym}(f)\,.$$

We have shown that Sym is indeed a functor from $\mathscr{B}$ to **Set**.[4]

Let X and Y be two finite sets and let f be a bijection from X to Y. Given a total order $\leq$ on X we get a total order $\leq_f$ on Y given by

$$f(x_1) \leq_f f(x_2) \iff x_1 \leq x_2$$

for all $x_1, x_2 \in X$. We define $\mathrm{Ord}(f)$ as the map

$$\mathrm{Ord}(f)\colon\ \mathrm{Ord}(X) \longrightarrow \mathrm{Ord}(Y)\,, \quad \leq \longmapsto \leq_f\,.$$

This defined a functor Ord from $\mathscr{B}$ to **Set**:

[4]The functor Sym lifts along the forgetful functor from **Grp** to **Set** to a functor from $\mathscr{B}$ to **Grp**.

- Let X be a set. For every total order $\leq$ on X, i.e., element of $\mathrm{Ord}(X)$, we have
$$x_1 \leq_{1_X} x_2 \iff 1_X(x_1) \leq_{1_X} 1_X(x_2) \iff x_1 \leq x_2$$
for all $x_1, x_2 \in X$, which means that the total order $\leq_{1_X} = \mathrm{Ord}(1_X)(\leq)$ coincides with the original order $\leq$. This shows that $\mathrm{Ord}(1_X) = 1_{\mathrm{Ord}(X)}$.
- Let
$$f : X \longrightarrow Y, \quad g : Y \longrightarrow Z$$
be two composable morphisms in $\mathscr{B}$. Given a total order $\leq$ on X, we have the chain of equivalences
$$\begin{aligned} & (g \circ f)(x_1) \, (\leq_f)_g \, (g \circ f)(x_2) \\ \iff & g(f(x_1)) \, (\leq_f)_g \, g(f(x_2)) \\ \iff & f(x_1) \leq_f f(x_2) \\ \iff & x_1 \leq x_2 \\ \iff & (g \circ f)(x_1) \leq_{g \circ f} (g \circ f)(x_2) \end{aligned}$$
for all $x_1, x_2 \in X$, which shows that $\mathrm{Ord}(g)(\mathrm{Ord}(f)(\leq)) = \mathrm{Ord}(g \circ f)(\leq)$. We have thus shown that
$$\mathrm{Ord}(g \circ f) = \mathrm{Ord}(g) \circ \mathrm{Ord}(f) .$$

This shows that Ord is indeed a functor from $\mathscr{B}$ to **Set**.

(b)

We have for every finite set X a distinguished element of $\mathrm{Sym}(X)$, namely the identity function 1_X. For every morphism
$$f : X \longrightarrow Y$$
in $\mathscr{B}$, we have
$$\mathrm{Sym}(f)(1_X) = f \circ 1_X \circ f^{-1} = f \circ f^{-1} = 1_Y .$$

This shows that the map $\mathrm{Sym}(f)$ carries the distinguished element of $\mathrm{Sym}(X)$ to the distinguished element of $\mathrm{Sym}(Y)$. In the following, we denote the distinguished element 1_X of $\mathrm{Sym}(X)$ by s_X.

Suppose that there exists a natural transformation α from Sym to Ord. For every finite set X we can then consider the element $t_X \coloneqq \alpha_X(s_X)$ of $\mathrm{Ord}(X)$. We have for every morphism

$$f\colon X \longrightarrow Y$$

in $\mathscr{B}$ from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Sym}(X) \arrow[r, "\mathrm{Sym}(f)"] \arrow[d, "\alpha_X"'] & \mathrm{Sym}(Y) \arrow[d, "\alpha_Y"] \\
\mathrm{Ord}(X) \arrow[r, "\mathrm{Ord}(f)"] & \mathrm{Ord}(Y)
\end{tikzcd}$$

It follows from the commutativity of this diagram that

$$\mathrm{Ord}(f)(t_X) = \mathrm{Ord}(f)(\alpha_X(s_X)) = \alpha_Y(\mathrm{Sym}(f)(s_X)) = \alpha_Y(s_Y) = t_Y\,.$$

This calculation tells us that the map $\mathrm{Ord}(f)$ carries the distinguished element t_X of $\mathrm{Ord}(X)$ to the distinguished element t_Y of $\mathrm{Ord}(Y)$.

However, such a choice of distinguished elements $(t_X)_{X \in \mathrm{Ob}(\mathscr{B})}$ cannot exist: If X is a finite set with at least two elements, then there exists a bijection f of X to itself that swaps these two elements. It then follows for every element $\leq$ of $\mathrm{Ord}(X)$ that $\mathrm{Ord}(f)(\leq)$ is distinct from $\leq$. Therefore, $\mathrm{Ord}(f)$ cannot carry t_X to t_X.

(c)

For a finite set X of cardinality n, both $\mathrm{Ord}(X)$ and $\mathrm{Sym}(X)$ have $n!$ elements.

Conclusion

For every finite set X, the two sets $\mathrm{Sym}(X)$ and $\mathrm{Ord}(X)$ have the same cardinality, and are therefore isomorphic as objects of **Set**. This means that the two functors Sym and Ord are pointwise isomorphic. However, we have seen above that there exists no natural transformation from Sym to Ord, and therefore in particular no natural isomorphism from Sym to Ord.

Exercise 1.3.32

(a)

There exists by assumption a functor G from $\mathscr{B}$ to $\mathscr{A}$ such that

$$G \circ F \cong 1_{\mathscr{A}} \,, \qquad F \circ G \cong 1_{\mathscr{B}} \,.$$

The isomorphism $F \circ G \cong 1_{\mathscr{B}}$ entails that

$$F(G(B)) \cong B$$

for every object B of $\mathscr{B}$. This shows that the functor F is essentially surjective.

To show that F is full and faithful let A and A' be two objects of $\mathscr{A}$. Let

$$\alpha : \; G \circ F \Longrightarrow 1_{\mathscr{A}}$$

be a natural isomorphism. The square diagram

$$\begin{tikzcd} GF(A) \arrow[r, "GF(f)"] \arrow[d, "\alpha_A"'] & GF(A') \arrow[d, "\alpha_{A'}"] \\ A \arrow[r, "f"] & A' \end{tikzcd}$$

commutes for every morphism f from A to A' in $\mathscr{A}$. We thus have

$$\alpha_{A'} \circ GF(f) \circ \alpha_A^{-1} = f$$

for every such morphism f. In other words, the composite

$$\begin{tikzcd} \mathscr{A}(A, A') \arrow[r, "F"] & \mathscr{B}(F(A), F(A')) \arrow[r, "G"] & \mathscr{A}(GF(A), GF(A')) \arrow[d, "\alpha_{A'} \circ (-) \circ \alpha_A^{-1}"] \\ & & \mathscr{A}(A, A') \end{tikzcd} \tag{1.2}$$

is the identity map. The first step of this composite, i.e., the map

$$\mathscr{A}(A, A') \xrightarrow{F} \mathscr{B}(F(A), F(A')) \,,$$

therefore is injective. This shows that the functor F is faithful.

Let us now show that the functor G is full. We have seen that the composite (1.2) is the identity map for any two objects A and A' of $\mathscr{A}$. The last step of this composite, i.e., the conjugation map $\alpha_{A'} \circ (-) \circ \alpha_A^{-1}$, is bijective, whence we find that the map

$$\mathscr{B}(F(A), F(A')) \xrightarrow{G} \mathscr{A}(GF(A), GF(A'))$$

is surjective. This tells us that the functor G is full when restricted to the image on F. But we need to show that the map

$$\mathscr{B}(B, B') \xrightarrow{G} \mathscr{A}(G(B), G(B'))$$

is surjective for *any* two objects B and B' of $\mathscr{B}$, and not just for those objects that lie in the image of F.

Thankfully, we already know that the functor F is essentially surjective. So every object of $\mathscr{B}$ lies in the image of F up to isomorphism. For the given two objects B and B' of $\mathscr{B}$ there hence exist two objects A and A' of $\mathscr{A}$ for which there exists isomorphisms

$$g : F(A) \longrightarrow B, \quad g' : F(A') \longrightarrow B'.$$

The square diagram

$$\begin{tikzcd}
\mathscr{B}(F(A), F(A')) \arrow[r, "g' \circ (-) \circ g^{-1}"] \arrow[d, "G"'] & \mathscr{B}(B, B') \arrow[d, "G"] \\
\mathscr{B}(GF(A), GF(A')) \arrow[r, "G(g') \circ (-) \circ G(g)^{-1}"] & \mathscr{B}(G(B), G(B'))
\end{tikzcd}$$

commutes by the functoriality of G. The two horizontal maps in this diagram are bijections because both g and g' are isomorphisms. We have seen above that the vertical map on the left-hand side is surjective. It follows that the vertical map on the right-hand side is again surjective, which shows the desired fullness.

We have thus shown that the functor G is full. By swapping the roles of F and G, we find that F is full.

We have altogether shown that equivalences are faithful, full, and essentially surjective.

(b)

Let F be a functor from $\mathscr{A}$ to $\mathscr{B}$ that is faithful, full, and essentially surjective. There exists for every object B of $\mathscr{B}$ an object $G(B)$ of $\mathscr{A}$ for which there exists an isomorphism

$$\varepsilon_B : FG(B) \longrightarrow B .$$

For every morphism $g : B \to B'$ in $\mathscr{B}$, the conjugate

$$\varepsilon_{B'}^{-1} \circ g \circ \varepsilon_B$$

is a morphism from $FG(B)$ to $FG(B')$). It follows that there exists a unique morphism $G(g)$ from $G(B)$ to $G(B')$ such that

$$FG(g) = \varepsilon_{B'}^{-1} \circ g \circ \varepsilon_B$$

because the functor F is both full (showing the existence of $G(g)$) and faithful (showing the uniqueness of $G(g)$).

The assignment G is a functor from $\mathscr{B}$ to $\mathscr{A}$:

- For every object B of $\mathscr{B}$, we have the equalities

  $$\varepsilon_B^{-1} \circ 1_B \circ \varepsilon_B = \varepsilon_B^{-1} \circ \varepsilon_B = 1_{FG(B)} = F(1_{G(B)}) .$$

  This shows that the morphism $1_{G(B)}$ satisfies the defining property of the morphism $G(1_B)$, so that

  $$G(1_B) = 1_{G(B)} .$$

- For any two composable morphisms

  $$g : B \longrightarrow B' , \quad g' : B' \longrightarrow B''$$

  in $\mathscr{B}$, we have the equalities

  $$\begin{aligned} \varepsilon_{B''}^{-1} \circ g' \circ g \circ \varepsilon_B &= \varepsilon_{B''}^{-1} \circ g' \circ \varepsilon_{B'} \circ \varepsilon_{B'}^{-1} \circ g \circ \varepsilon_B \\ &= F(G(g')) \circ F(G(g)) \\ &= F(G(g') \circ G(g)) . \end{aligned}$$

  This shows that the composite $G(g') \circ G(g)$ satisfies the defining property of the morphism $G(g' \circ g)$, so that

  $$G(g' \circ g) = G(g') \circ G(g) .$$

This shows that the assignment G is indeed a functor.

The family $\varepsilon = (\varepsilon_B)_{B \in \mathrm{Ob}(\mathscr{B})}$ is (by construction) a natural isomorphism from the functor FG to the functor $1_{\mathscr{B}}$. In the following, we will construct a natural isomorphism α from GF to $1_{\mathscr{A}}$.

We have for every object A of $\mathscr{A}$ the morphism

$$\varepsilon_{F(A)} : FGF(A) \longrightarrow F(A)$$

in $\mathscr{B}$. It follows that there exists a unique morphism

$$\alpha_A : GF(A) \longrightarrow A$$

in $\mathscr{A}$ such that $\varepsilon_{F(A)} = F(\alpha_A)$, because the functor F is both full (showing the existence of α_A) and faithful (showing the uniqueness of α_A). The morphism α_A is an isomorphism because the functor F is full and faithful and therefore reflects isomorphisms by the following lemma.

Lemma 1.B. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories and let F be a functor from $\mathscr{A}$ to $\mathscr{B}$ that is both full and faithful. Let f be a morphism in $\mathscr{A}$ such that $F(f)$ is an isomorphism. Then f is an isomorphism.

Proof. Let A be the domain of the morphism f, and let A' be its codomain. We have in $\mathscr{B}$ the morphism $F(f)^{-1}$ from $F(A')$ to $F(A)$. There exists a morphism f' from A' to A with

$$F(f') = F(f)^{-1}$$

because the functor F is full. We find for the morphism $f' \circ f$ from A to A that

$$F(f' \circ f) = F(f') \circ F(f) = F(f)^{-1} \circ F(f) = 1_{F(A)} = F(1_A) .$$

It follows that $f' \circ f = 1_A$ because the functor F is faithful. We find in the same way that also $f \circ f' = 1_{A'}$. We have thus shown that the morphism f is an isomorphism with inverse given by f'. ∎

The resulting family $\alpha = (\alpha_A)_{A \in \mathrm{Ob}(\mathscr{A})}$ is a natural isomorphism from GF to A. We still need to prove the naturality of α. To this end we need to check that for every morphism

$$f : A \longrightarrow A'$$

in $\mathscr{A}$, the following square diagram commutes:

$$\begin{tikzcd} GF(A) \arrow[r, "GF(f)"] \arrow[d, "\alpha_A"'] & GF(A') \arrow[d, "\alpha_{A'}"] \\ A \arrow[r, "f"] & A' \end{tikzcd}$$

It suffices to show that this diagram commutes after we apply the functor F to it, because F is faithful. We hence need to show that the following square diagram commutes:

$$\begin{tikzcd} FGF(A) \arrow[r, "FGF(f)"] \arrow[d, "\varepsilon_{F(A)}"'] & FGF(A') \arrow[d, "\varepsilon_{F(A')}"] \\ F(A) \arrow[r, "F(f)"] & F(A') \end{tikzcd}$$

This desired commutativity follows from the naturality of ε.

The existence of the natural isomorphisms

$$\varepsilon : FG \Longrightarrow 1_{\mathscr{B}}, \quad \alpha : GF \Longrightarrow 1_{\mathscr{A}}$$

shows that the two functors F and G form an equivalence between the categories $\mathscr{A}$ and $\mathscr{B}$. This entails that F is an equivalence of categories.

Exercise 1.3.33

The composition of morphisms in **Mat** is given by matrix multiplication.

We consider an auxiliary category $\mathscr{B}$ of "vector spaces together with bases". This category is defined as follows: The objects of $\mathscr{B}$ are pairs (V, B) consisting of a finite-dimensional $\Bbbk$-vector space V and a basis B of V. For any two objects (V, B) and (W, C) of $\mathscr{B}$, the respective set of morphism $\mathscr{B}((V, B), (W, C))$ is simply the set of linear maps from V to W. The composition of morphisms in $\mathscr{B}$ is the usual composition of linear maps.

We have a forgetful functor U from $\mathscr{B}$ to **FDVect** given by

$$U((V, B)) := V \quad \text{and} \quad U(f) := f$$

on objects and morphisms respectively. The functor U is full, faithful and surjective, and hence an equivalence of categories.

We also have a functor F from $\mathscr{B}$ to **Mat** as follows: For every object (V, B) of $\mathscr{B}$, the action of F on (V, B) is given by

$$F((V, B)) := \dim(V) .$$

For every morphism

$$f : (V, B) \longrightarrow (W, C)$$

in $\mathscr{B}$, we choose $F(f)$ as the matrix that represents the linear map f with respect to the bases B and C of V and W. The functor F is full, faithful and surjective, and is therefore an equivalence of categories.

The category $\mathscr{B}$ is both equivalent to **FDVect** and to **Mat**. It follows (from the upcoming Exercise 1.3.34) that **FDVect** and **Mat** are also equivalent.

An essential inverse F' of the functor F can explicitly be constructed on objects by $F'(n) = (\Bbbk^n, (e_1, \dots, e_n))$ for every natural number n, and on morphisms by $F'(A)(x) = Ax$ for every matrix A and column vector x (of suitable sizes).

An essential inverse U' of the functor U together with a natural isomorphism from $U' \circ U$ to $1_{\mathbf{FDVect}}$ amounts to choosing a basis for every finite-dimensional $\Bbbk$-vector space.

The composite $U \circ F'$ is an equivalence of categories from **Mat** to **FDVect**$\Bbbk$. It assigns to each natural number n the vector space $\Bbbk^n$, and regards every matrix of size $m \times n$ as a linear map from $\Bbbk^n$ to $\Bbbk^m$ via matrix-vector multiplication.

We might regard the functor $U \circ F'$ as a "canonical" functor from **Mat** to **FDVect**$\Bbbk$. The functor $F \circ U'$, on the other hand, is not "canonical" since it depends on choosing bases.

Exercise 1.3.34

Every category is equivalent – even isomorphic – to itself via its identity functor. This shows that equivalence of categories is reflexive.

Two categories $\mathscr{A}$ and $\mathscr{B}$ are equivalent if and only if there exist functors

$$F : \mathscr{A} \longrightarrow \mathscr{B} , \quad F' : \mathscr{B} \longrightarrow \mathscr{A}$$

such that $F' \circ F \cong 1_{\mathscr{A}}$ and $F \circ F' \cong 1_{\mathscr{B}}$. This condition is symmetric in $\mathscr{A}$ and $\mathscr{B}$, whence equivalence of categories is symmetric.

Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be three categories such that $\mathscr{A}$ is equivalent to $\mathscr{B}$ and $\mathscr{B}$ is equivalent to $\mathscr{C}$. This means that there exist functors

$$F : \mathscr{A} \longrightarrow \mathscr{B}\,, \quad F' : \mathscr{B} \longrightarrow \mathscr{A}\,, \quad G : \mathscr{B} \longrightarrow \mathscr{C}\,, \quad G' : \mathscr{C} \longrightarrow \mathscr{B}$$

such that

$$F' \circ F \cong 1_{\mathscr{A}}\,, \quad F \circ F' \cong 1_{\mathscr{B}}\,, \quad G' \circ G \cong 1_{\mathscr{B}}\,, \quad G \circ G' \cong 1_{\mathscr{C}}\,.$$

It follows that

$$(G \circ F) \circ (F' \circ G') = G \circ F \circ F' \circ G' \cong G \circ 1_{\mathscr{B}} \circ G' = G \circ G' \cong 1_{\mathscr{C}}$$

and similarly

$$(F' \circ G') \circ (G \circ F) = F' \circ G' \circ G \circ F \cong F' \circ 1_{\mathscr{B}} \circ F = F' \circ F \cong 1_{\mathscr{A}}\,.$$

This shows that the functor $G \circ F$ is again an equivalence of categories, with essential inverse given by $F' \circ G'$. We have thus shown that equivalence of categories is transitive.

Chapter 2

Adjoints

2.1 Definition and examples

Exercise 2.1.12

Adjoint functors

- Let $\Bbbk$ be a field and let V be a $\Bbbk$-vector space. The two functors

  $$(-) \otimes_{\Bbbk} V : \mathbf{Vect}_{\Bbbk} \longrightarrow \mathbf{Vect}_{\Bbbk} \quad \text{and} \quad \mathrm{Hom}_{\Bbbk}(V, -) : \mathbf{Vect}_{\Bbbk} \longrightarrow \mathbf{Vect}_{\Bbbk}$$

  are adjoint, with $(-) \otimes_{\Bbbk} V$ being left adjoint to $\mathrm{Hom}_{\Bbbk}(V, -)$. (This is an example in the vain of Example 2.1.16.)

- Let more generally R and S be two rings and let M be an R-S-bimodule. The two functors

  $$(-) \otimes_R M : \mathbf{Mod}\text{-}R \longrightarrow \mathbf{Mod}\text{-}S \quad \text{and} \quad \mathrm{Hom}_S(M, -) : \mathbf{Mod}\text{-}S \longrightarrow \mathbf{Mod}\text{-}R$$

  are adjoint, with $(-) \otimes_R M$ being left adjoint to $\mathrm{Hom}_S(M, -)$.

- For every category $\mathscr{A}$ let $U(\mathscr{A})$ be its underlying graph, and for every functor F from a category $\mathscr{A}$ to a category $\mathscr{B}$ let $U(F)$ be its induced homomorphism of graphs from $U(\mathscr{A})$ to $U(\mathscr{B})$. This assignment U is a functor from **Cat** to **Graph**, the category of graphs.[1]

  For every graph Γ let $P(\Gamma)$ be the following category: The objects of $P(\Gamma)$ are the vertices of Γ. For any two vertices x and y of Γ, the morphisms

[1] By a graph we mean a directed graph, possibly with parallel edges as well as loops. We also impose no finiteness conditions on the graphs under consideration.

from x to y in $P(\Gamma)$ are precisely the paths from x to y in Γ.[2] The composition of morphisms of $P(\Gamma)$ is the composition of paths in Γ.[3] The category $P(\Gamma)$ is the **path category** of Γ.

Every homomorphism of graphs f from Γ to Γ' induces a functor $P(f)$ from $P(\Gamma)$ to $P(\Gamma')$, given on objects by $P(f)(x) = f(x)$ for every vertex x of Γ, and on morphisms by

$$P(f)((x, \alpha_1, \dots, \alpha_n, y)) = (f(x), f(\alpha_1), \dots, f(\alpha_n), f(y))$$

for every path $(x, \alpha_1, \dots, \alpha_n, y)$ in Γ. We thus arrive at a functor P from **Graph** to **Cat**.

The functors P and U are adjoint, with P being left adjoint to U. (We may regard U as a forgetful functor, and $P(\Gamma)$ as the "free category on Γ".)

Initial objects

- The empty category is initial in **CAT**.
- Let R be a ring. The zero module is initial in R-**Mod**.
- The initial objects in the category of pointed sets are precisely the one-element pointed sets.
- Let P be a partially ordered set and let $\mathscr{P}$ be the corresponding category. An initial object of $\mathscr{P}$ is the same as a least element of P.

Terminal objects

- Let R be a ring. The zero module is terminal in R-**Mod**.
- Let P be a partially ordered set and let $\mathscr{P}$ be its corresponding category. A terminal object of $\mathscr{P}$ is the same as a greatest element of P.
- The terminal objects in **Top** are precisely those topological spaces that consist of only a single point.

[2] By a path from x to y in Γ we mean a tuple $p = (x, \alpha_1, \dots, \alpha_n, y)$ of the following form: $\alpha_1, \dots, \alpha_n$ are edges in Γ such that α_1 starts in x, the end vertex of α_i is the start vertex of α_{i+1} for all $i = 1, \dots, n-1$, and α_n ends in y. The vertex x is the start vertex of p, the vertex y is the end vertex of p, and the number n is the length of p. This entails in particular that there exists for every vertex x of Γ a unique path of length 0 from x to x in Γ, given by the tuple (x, x). We emphasize that for any two distinct vertices x and y of Γ their associated paths of length 0 are again distinct.

[3] The composite $q \circ p$ of two paths $p = (x, \alpha_1, \dots, \alpha_n, y)$ and $q = (y, \alpha_{n+1}, \dots, \alpha_m, z)$ is the path $(x, \alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_m, z)$.

Exercise 2.1.13

Let $\mathscr{A}$ and $\mathscr{B}$ be two discrete categories with underlying classes of objects A and B.

We consider first two functors

$$F : \mathscr{A} \longrightarrow \mathscr{B}, \quad G : \mathscr{B} \longrightarrow \mathscr{A}$$

such that F is left adjoint to G. We may regard these two functors as functions

$$f : A \longrightarrow B, \quad g : B \longrightarrow A.$$

For every element a of A, we have

$$\mathscr{A}(a, g(f(a))) = \mathscr{A}(a, GF(a)) \cong \mathscr{B}(F(a), F(a)) \neq \varnothing.$$

This means that $g(f(a)) = a$ because the category $\mathscr{A}$ is discrete. We find in the same way that $f(g(b)) = b$ for every element b of B. The two functions f and g are therefore mutually inverse bijections.

Suppose on the other hand that

$$f : A \longrightarrow B, \quad g : B \longrightarrow A$$

are two mutually inverse bijections. We may regard these functions as mutually inverse isomorphisms of categories

$$F : \mathscr{A} \longrightarrow \mathscr{B}, \quad G : \mathscr{B} \longrightarrow \mathscr{A}.$$

We then have

$$\mathscr{B}(F(a), b) = \mathscr{B}(F(a), F(G(b))) \cong \mathscr{A}(a, G(b))$$

for every element a of A and every element b of B. This bijection is natural in both a and b because the only morphisms in $\mathscr{A}$ and in $\mathscr{B}$ are the identity morphisms. This shows that the functors F and G are adjoint, with F being left adjoint to G.

We have thus shown that an adjunction between two discrete categories $\mathscr{A}$ and $\mathscr{B}$ is the same as a pair of mutually inverse bijections between their underlying classes of objects.

Exercise 2.1.14

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories and let

$$F \colon \mathscr{A} \longrightarrow \mathscr{B}\,, \quad G \colon \mathscr{B} \longrightarrow \mathscr{A}$$

be two functors.

Suppose first that the two naturality equations (2.2) and (2.3) are satisfied. For all morphisms in $\mathscr{A}$ of the form

$$A' \xrightarrow{\;p\;} A \xrightarrow{\;f\;} G(B) \xrightarrow{\;G(q)\;} G(B')\,,$$

we then have the chain of equalities

$$\begin{aligned}
&\overline{\left(A' \xrightarrow{\;p\;} A \xrightarrow{\;f\;} G(B) \xrightarrow{\;G(q)\;} G(B')\right)} \\
&= \overline{\left(A' \xrightarrow{\;fp\;} G(B) \xrightarrow{\;G(q)\;} G(B')\right)} \\
&= \overline{\left(A' \xrightarrow{\;\overline{\overline{fp}}\;} G(B) \xrightarrow{\;G(q)\;} G(B')\right)} \\
&= \left(F(A') \xrightarrow{\;\overline{fp}\;} B \xrightarrow{\;q\;} B'\right) && \text{(by (2.2))} \\
&= \left(\overline{\left(A' \xrightarrow{\;p\;} A \xrightarrow{\;f\;} G(B)\right)} \xrightarrow{\;q\;} B'\right) \\
&= \left(F(A') \xrightarrow{\;F(p)\;} F(A) \xrightarrow{\;\overline{f}\;} B \xrightarrow{\;q\;} B'\right). && \text{(by (2.3))}
\end{aligned}$$

This shows that the given naturality equation holds.

Suppose now conversely that the given naturality equation holds. Then

$$\begin{aligned}
\overline{\left(F(A) \xrightarrow{\;g\;} B \xrightarrow{\;q\;} B'\right)} &= \overline{\left(F(A) \xrightarrow{\;1_{F(A)}\;} F(A) \xrightarrow{\;g\;} B \xrightarrow{\;q\;} B'\right)} \\
&= \overline{\left(F(A) \xrightarrow{\;F(1_A)\;} F(A) \xrightarrow{\;\overline{\overline{g}}\;} B \xrightarrow{\;q\;} B'\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(A \xrightarrow{1_A} A \xrightarrow{\overline{g}} B \xrightarrow{G(q)} B'\right) \\
&= \left(A \xrightarrow{\overline{g}} B \xrightarrow{G(q)} B'\right),
\end{aligned}
$$

which shows that the naturality equation (2.2). We find similarly that

$$
\begin{aligned}
\overline{\left(A' \xrightarrow{p} A \xrightarrow{f} G(B)\right)} &= \overline{\left(A' \xrightarrow{p} A \xrightarrow{f} G(B) \xrightarrow{1_{G(B)}} G(B)\right)} \\
&= \overline{\left(A' \xrightarrow{p} A \xrightarrow{f} G(B) \xrightarrow{G(1_B)} G(B)\right)} \\
&= \left(F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\overline{f}} B \xrightarrow{1_B} B\right) \\
&= \left(F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\overline{f}} B\right),
\end{aligned}
$$

which shows the naturality equation (2.3).

Exercise 2.1.15

We have for every object B of $\mathscr{B}$ the bijections

$$\mathscr{B}(F(I), B) \cong \mathscr{A}(I, G(B)) \cong \{*\}.$$

This means that there exists for every object B of $\mathscr{B}$ a unique morphism from $F(I)$ to B in $\mathscr{B}$. The object $F(I)$ is therefore initial in $\mathscr{B}$.

Let T be a terminal object of $\mathscr{B}$. We have for every object A of $\mathscr{A}$ the bijections

$$\mathscr{A}(A, G(T)) \cong \mathscr{B}(F(A), T) \cong \{*\}.$$

This means that there exists for every object A of $\mathscr{A}$ a unique morphism from A to $G(T)$ in $\mathscr{A}$. The object $G(T)$ is therefore terminal in $\mathscr{A}$.

Exercise 2.1.16

(b)

We observe that the functor category $[G, \mathbf{Vect}_{\Bbbk}]$ is isomorphic to the module category $\Bbbk[G]$-**Mod**.

Suppose that H is another group and let φ be a homomorphism of groups from H to G. We may regard φ as a functor from H to G. We get an induced functor

$$\varphi^* : [G, \mathbf{Vect}_{\Bbbk}] \longrightarrow [H, \mathbf{Vect}_{\Bbbk}]$$

that is given by pre-composition with φ. The homomorphism of groups φ also induces a homomorphism of $\Bbbk$-algebras

$$\Bbbk[\varphi] : \Bbbk[G] \longrightarrow \Bbbk[H] .$$

Under the isomorphism of $[G, \mathbf{Vect}_{\Bbbk}]$ with $\Bbbk[G]$-**Mod** and the isomorphism of $[H, \mathbf{Vect}_{\Bbbk}]$ with $\Bbbk[H]$-**Mod**, the above functor φ^* corresponds to the restriction functor

$$\mathrm{Res}^G_H : \Bbbk[G]\text{-}\mathbf{Mod} \longrightarrow \Bbbk[H]\text{-}\mathbf{Mod}$$

induced by $\Bbbk[\varphi]$. This restriction functor admits both a left adjoint and a right adjoint. A left adjoint is given by

$$\mathrm{Ind}^H_G := \Bbbk[G] \otimes_{\Bbbk[H]} (-) : \Bbbk[H]\text{-}\mathbf{Mod} \longrightarrow \Bbbk[G]\text{-}\mathbf{Mod}$$

and a right adjoint is given by

$$\mathrm{Coind}^H_G := \mathrm{Hom}_{\Bbbk[H]}(\Bbbk[G], -) : \Bbbk[H]\text{-}\mathbf{Mod} \longrightarrow \Bbbk[G]\text{-}\mathbf{Mod} .$$

We may identify the category $\mathbf{Vect}_{\Bbbk}$ with $\Bbbk[1]$-**Mod** (where 1 denotes the trivial group). We then find from the above discussion that the unique homomorphisms of groups from 1 to G induces adjoint functors

$$F \dashv U \dashv C ,$$

given by the forgetful functor

$$U : \Bbbk[G]\text{-}\mathbf{Mod} \longrightarrow \mathbf{Vect}_{\Bbbk} ,$$

the extension of scalars

$$F := \Bbbk[G] \otimes_{\Bbbk} (-) : \mathbf{Vect}_{\Bbbk} \longrightarrow \Bbbk[G]\text{-}\mathbf{Mod} ,$$

and

$$C := \mathrm{Hom}_{\Bbbk}(\Bbbk[G], -) : \mathbf{Vect}_{\Bbbk} \longrightarrow \Bbbk[G]\text{-}\mathbf{Mod} .$$

Similarly, we can consider the unique homomorphism of groups from G to 1. This homomorphism induces adjunctions

$$C \dashv T \dashv I .$$

The functor

$$T : \mathbf{Vect}_{\Bbbk} \longrightarrow \Bbbk[G]\text{-}\mathbf{Mod}$$

regards a vector space as a trivial $\Bbbk[G]$-module (i.e., as a trivial representation of G); the functor

$$I : \Bbbk[G]\text{-}\mathbf{Mod} \longrightarrow \mathbf{Vect}_{\Bbbk}$$

can be described as

$$I = \operatorname{Hom}_{\Bbbk[G]}(\Bbbk, -) \cong (-)^G ,$$

assigning to each $\Bbbk[G]$-module its linear subspace of invariants; the functor

$$C : \Bbbk[G]\text{-}\mathbf{Mod} \longrightarrow \mathbf{Vect}_{\Bbbk}$$

can be described as

$$C = \Bbbk \otimes_{\Bbbk[G]} (-) \cong (-)_G ,$$

assigning to each $\Bbbk[G]$-module M its linear quotient of coinvariants, i.e.,

$$C(M) \cong M_G = M/\langle m - gm \mid g \in G, m \in M \rangle_{\Bbbk} .$$

(a)

We can proceed as in part (b) of this exercise. For this, we think about the functor category $[G, \mathbf{Set}]$ as the category of G-sets, which we shall denote by G-**Set**. We also think about **Set** as 1-**Set**, where 1 denotes the trivial group.

Let H be another group and let φ be a homomorphism of groups from H to G. We regard φ as a functor from H to G and get an induced functor

$$\varphi^* : [G, \mathbf{Set}] \longrightarrow [H, \mathbf{Set}] .$$

This functor corresponds to the restriction functor

$$\operatorname{Res}^G_H : G\text{-}\mathbf{Set} \longrightarrow H\text{-}\mathbf{Set} .$$

This restriction functor admits both a left adjoint and a right adjoint. A left adjoint is given by

$$\mathrm{Ind}_G^H \coloneqq G \times_H (-) \colon H\text{-}\mathbf{Set} \longrightarrow G\text{-}\mathbf{Set}\,,$$ [4]

and a right adjoint is given by

$$\mathrm{Coind}_G^H \coloneqq \mathrm{Hom}_H(G, -)\,. \colon H\text{-}\mathbf{Set} \longrightarrow G\text{-}\mathbf{Set}\,.$$

The unique homomorphism of groups from 1 to G induces adjoint functors

$$F \dashv U \dashv C\,,$$

where

$$U \colon G\text{-}\mathbf{Set} \longrightarrow \mathbf{Set}$$

is the forgetful functor. A left adjoint is given by the functor

$$F \coloneqq G \times (-) \colon \mathbf{Set} \longrightarrow G\text{-}\mathbf{Set}\,,$$

and a right adjoint is given by the functor

$$C \coloneqq \mathrm{Map}(G, -) \colon \mathbf{Set} \longrightarrow G\text{-}\mathbf{Set}\,.$$

(We can also describe F and C as $F(X) = \coprod_{g \in G} X$ and $C(X) = \prod_{g \in G} X$. The action of G on these sets is then given by permutation of the summands, respectively factors.)

The unique homomorphism of groups from G to 1 does similarly induce adjoint functors

$$O \dashv T \dashv I\,.$$

The functor

$$T \colon \mathbf{Set} \longrightarrow G\text{-}\mathbf{Set}$$

regard each set as trivial G-sets; its right adjoint

$$I \colon G\text{-}\mathbf{Set} \longrightarrow \mathbf{Set}$$

[4] For any H-set X, the G-set $G \times_H X$ is given by the set $(G \times X)/\sim$ where $\sim$ is the equivalence relation generated by $(gh, x) \sim (g, hx)$, and the action of G on $G \times_H X$ is given by $g' \cdot [(g, x)] = [(g'g, x)]$.

can be described as

$$I = \mathrm{Hom}_G(1, -) \cong (-)^G,$$

assigning to each G-set its subset of invariants; the right adjoint

$$O : G\text{-}\mathbf{Set} \longrightarrow \mathbf{Set}$$

can be described as

$$O = 1 \times_G (-) \cong (-)/G,$$

assigning to each G-set its set of orbits.

Exercise 2.1.17

The category $\mathcal{O}(X)$ of open subsets of X admits a unique initial object, namely the empty subset $\varnothing$, as well as a unique terminal object, namely the entire space X. Instead of $\mathcal{O}(X)$ we will consider an arbitrary category $\mathscr{A}$ that admits an initial object $I(\mathscr{A})$ and a terminal object $T(\mathscr{A})$, and which satisfies the following additional properties: there exist no morphism from $T(\mathscr{A})$ to any non-terminal object of $\mathscr{A}$, and there exists dually no morphism from any non-initial object of $\mathscr{A}$ to $I(\mathscr{A})$.[5]

The category **Set** also admits a unique initial object, namely the empty set $\varnothing$, as well as terminal objects, namely the one-element sets. Instead of **Set** we will consider an arbitrary category $\mathscr{B}$ that admit an initial object $I(\mathscr{B})$ and a terminal object $T(\mathscr{B})$.

The functor Δ

We start off with the functor

$$\Delta : \mathscr{B} \longrightarrow [\mathscr{A}^{\mathrm{op}}, \mathscr{B}]$$

that assigns to each object B of $\mathscr{B}$ the constant functor at B. To each morphism

$$g : B \longrightarrow B'$$

in $\mathscr{B}$ it assigns the natural transformation $\Delta(g)$ from $\Delta(B)$ to $\Delta(B')$ given by the component

$$\Delta(g)_A := g$$

[5]We use the notations $T(\mathscr{A})$ and $I(\mathscr{A})$ instead of the more appropriate $T_{\mathscr{A}}$ and $I_{\mathscr{A}}$ for better readability.

for every object A of $\mathscr{A}$. The assignment Δ is indeed functorial:

- We have for every object B of $\mathscr{B}$ the equalities

$$\Delta(1_B)_A = 1_B = 1_{\Delta(B)(A)} = (1_{\Delta(B)})_A$$

  for every object A of $\mathscr{A}$, and therefore the equality

$$\Delta(1_B) = 1_{\Delta(B)}\,.$$

- We have for every two composable morphisms

$$g : B \longrightarrow B'\,, \quad g' : B' \longrightarrow B''$$

  in $\mathscr{B}$ the equalities

$$\Delta(g' \circ g)_A = g' \circ g = \Delta(g')_A \circ \Delta(g)_A = (\Delta(g') \circ \Delta(g))_A$$

  for every object A of $\mathscr{A}$, and therefore the equality

$$\Delta(g' \circ g) = \Delta(g') \circ \Delta(g)\,.$$

The functor Γ

Let F be a contravariant functor from $\mathscr{A}$ to $\mathscr{B}$. A natural transformation β from $\Delta(B)$ to F is a family $(\beta_A)_{A \in \mathrm{Ob}(\mathscr{A})}$ of morphisms β_A from $\Delta(B)(A)$ to $F(A)$ such that for every morphism

$$f : A \longrightarrow A'$$

in $\mathscr{A}$, the resulting diagram

$$\begin{tikzcd}
\Delta(B)(A') \arrow[r, "\beta_{A'}"] \arrow[d, "1_B"'] & F(A') \arrow[d, "F(f)"] \\
\Delta(B)(A) \arrow[r, "\beta_A"'] & F(A)
\end{tikzcd}$$

in $\mathscr{B}$ commutes. By using the definition of Δ, this diagram can equivalently be expressed in the following triangular form:

$$\begin{tikzcd} & F(A') \arrow[dd, "F(f)"] \\ B \arrow[ur, "\beta_{A'}"] \arrow[dr, "\beta_A"'] & \\ & F(A) \end{tikzcd}$$

We may consider for the object A' the terminal object $T(\mathscr{A})$, and consequently for the morphism f the unique morphism from A to $T(\mathscr{A})$. We then find from the commutativity of the above triangular diagram that the natural transformation β is uniquely determined by its component at $T(\mathscr{A})$, i.e., by the morphism

$$\beta_{T(\mathscr{A})} \colon B \longrightarrow F(T(\mathscr{A}))\,.$$

In other words, the map

$$\overline{(-)} \colon [\mathscr{A}^{\mathrm{op}}, \mathscr{B}](\Delta(B), F) \longrightarrow \mathscr{B}(B, F(T(\mathscr{A})))\,, \quad \beta \longmapsto \beta_{T(\mathscr{A})} \tag{2.1}$$

is injective. Let us show that it is also surjective.

Let g be an arbitrary morphism in $\mathscr{B}$ from B to $F(T(\mathscr{A}))$. For every object A of $\mathscr{A}$ let t_A be the unique morphisms from A to the terminal object $T(\mathscr{A})$, and let β_A be the resulting morphisms in $\mathscr{B}$ given by

$$\beta_A \colon B \xrightarrow{g} F(T(\mathscr{A})) \xrightarrow{F(t_A)} F(A)\,.$$

The resulting tuple $\beta \coloneqq (\beta_A)_{A \in \mathrm{Ob}(\mathscr{A})}$ is a natural transformation from $\Delta(B)$ to F. Indeed, given any morphisms

$$f \colon A \longrightarrow A'$$

in $\mathscr{A}$, we have the commutative diagram

$$\begin{tikzcd} & T(\mathscr{A}) & \\ A \arrow[ur, "t_A"] \arrow[rr, "f"] & & A' \arrow[ul, "t_{A'}"'] \end{tikzcd}$$

in $\mathscr{A}$ because the object $T(\mathscr{A})$ is terminal in $\mathscr{A}$. We conclude that the following diagram in $\mathscr{B}$ is again commutative:

$$\begin{tikzcd}
 & B \arrow[d, "g"] \arrow[ddl, bend right, "\beta_A"'] \arrow[ddr, bend left, "\beta_{A'}"] & \\
 & F(T(\mathscr{A})) \arrow[dl, "F(t_A)"'] \arrow[dr, "F(t_{A'})"] & \\
F(A) & & F(A') \arrow[ll, "F(f)"']
\end{tikzcd}$$

The commutativity of the outer diagram

$$\begin{tikzcd}
 & B \arrow[dl, "\beta_A"'] \arrow[dr, "\beta_{A'}"] & \\
F(A) & & F(A') \arrow[ll, "F(f)"']
\end{tikzcd}$$

shows the claimed naturality of β.

We note that $t_{T(\mathscr{A})} = 1_{T(\mathscr{A})}$, and that therefore

$$\beta_{T(\mathscr{A})} = F(t_{T(\mathscr{A})}) \circ g = F(1_{T(\mathscr{A})}) \circ g = 1_{F(T(\mathscr{A}))} \circ g = g\,.$$

We have therefore shown the surjectivity of (2.1), and thus altogether the bijectivity of (2.1).

Motivated by this bijection, we choose Γ as the evaluation functor at $T(\mathscr{A})$ [6] (We talk about evaluation at $T(\mathscr{A})$ because we view $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}]$ as the category of contravariant functors from $\mathscr{A}$ to $\mathscr{B}$. If we regard $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}]$ as the category of covariant functor from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}$, then Γ is the evaluation functor at $I(\mathscr{A}^{\mathrm{op}})$.) We can then rewrite (2.1) as the bijection

$$\overline{(-)} :\; [\mathscr{A}^{\mathrm{op}}, \mathscr{B}](\Delta(B), F) \longrightarrow \mathscr{B}(B, \Gamma(F))\,, \quad \beta \longmapsto \beta_{T(\mathscr{A})}\,. \tag{2.2}$$

[6] If we consider for $\mathscr{A}$ and $\mathscr{B}$ the categories $\mathcal{O}(X)$ and **Set**, then $\Gamma(F)$ is given by $F(X)$. This means that the functor Γ assigns to each presheaf its global sections. The chosen letter Γ is a reference to these global sections.

It remains to check that this bijection is natural, since this will then show that the functor Γ is right adjoint to the functor Δ.

We first check the naturality of (2.2) in the object B of $\mathscr{B}$. For this, we have to check that for every morphism

$$g : B \longrightarrow B'$$

in $\mathscr{B}$ the following diagram commutes:

$$\begin{tikzcd}
{[\mathscr{A}^{\mathrm{op}}, \mathscr{B}](\Delta(B), F)} \arrow[r, "\overline{(-)}"] & \mathscr{B}(B, \Gamma(F)) \\
{[\mathscr{A}^{\mathrm{op}}, \mathscr{B}](\Delta(B'), F)} \arrow[r, "\overline{(-)}"] \arrow[u, "\Delta(g)^*"] & \mathscr{B}(B', \Gamma(F)) \arrow[u, "g^*"']
\end{tikzcd}$$

This diagram indeed commutes, because we have for every element β of the bottom-left corner the chain of equalities

$$\begin{aligned}
\overline{\Delta(g)^*(\beta)} &= \overline{\beta \circ \Delta(g)} \\
&= (\beta \circ \Delta(g))_{T(\mathscr{A})} \\
&= \beta_{T(\mathscr{A})} \circ \Delta(g)_{T(\mathscr{A})} \\
&= \beta_{T(\mathscr{A})} \circ g \\
&= \overline{\beta} \circ g \\
&= g^*(\overline{\beta}) \,.
\end{aligned}$$

The bijection $\overline{(-)}$ is also natural in F. To prove this, we need to check that for every morphism

$$\gamma : F \Longrightarrow F'$$

in $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}]$ the following diagram commutes:

$$\begin{tikzcd}
{[\mathscr{A}^{\mathrm{op}}, \mathscr{B}](\Delta(B), F)} \arrow[r, "\overline{(-)}"] \arrow[d, "\gamma_*"'] & \mathscr{B}(B, \Gamma(F)) \arrow[d, "\Gamma(\gamma)_*"] \\
{[\mathscr{A}^{\mathrm{op}}, \mathscr{B}](\Delta(B), F')} \arrow[r, "\overline{(-)}"] & \mathscr{B}(B, \Gamma(F'))
\end{tikzcd}$$

This diagram commutes because for every element β of the top-left corner we have the equalities

$$\overline{\gamma_*(\beta)} = \overline{\gamma \circ \beta} = (\gamma \circ \beta)_{T(\mathscr{A})} = \gamma_{T(\mathscr{A})} \circ \beta_{T(\mathscr{A})} = \Gamma(\gamma) \circ \overline{\beta} = \Gamma(\gamma)_*(\overline{\beta}) \,.$$

We have thus altogether constructed a functor Γ from $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}]$ to $\mathscr{B}$ that is right adjoint to the previous functor Δ.

The functor ∇

Let us first try to motivate the definition of ∇.

Let F be a contravariant functor from $\mathscr{A}$ to $\mathscr{B}$, and let B be an object of $\mathscr{B}$. To construct the desired right adjoint functor ∇ of Γ we need to “extend” this object B to a contravariant functor $\nabla(B)$ from $\mathscr{A}$ to $\mathscr{B}$. This needs to be done in such a way that natural transformations from F to $\nabla(B)$ are “the same” as morphisms in $\mathscr{B}$ from $\Gamma(F) = F(T(\mathscr{A}))$ to B.

In other words, we need to extend morphisms from $F(T(\mathscr{A}))$ to B into natural transformations from F to a suitable functor $\nabla(B)$.

For any such a natural transformation β, its component $\beta_{T(\mathscr{A})}$ is a morphism from $F(T(\mathscr{A}))$ to $\nabla(B)(T(\mathscr{A}))$. We would therefore like to choose $\nabla(B)(T(\mathscr{A}))$ as B, in the hope of making the map

$$[\mathscr{A}^{\mathrm{op}}, \mathscr{B}](F, \nabla(B)) \longrightarrow \mathscr{B}(\Gamma(F), B) \,, \quad \beta \longmapsto \beta_{T(\mathscr{A})}$$

a bijection. To ensure this bijectivity, we then want all other components of β to be “trivial” in a suitable sense. We will be able to achieve this by choosing $\nabla(B)(A)$ as the terminal object of $\mathscr{B}$ whenever A is (essentially) distinct from $T(\mathscr{A})$.

Motivated by the above discussion, we set

$$\nabla(B)(A) \coloneqq \begin{cases} B & \text{if } A \text{ is terminal in } \mathscr{A}, \\ T(\mathscr{B}) & \text{otherwise,} \end{cases}$$

for every object A of $\mathscr{A}$.[7] To define the action of $\nabla(B)$ on morphisms, we consider an arbitrary morphism

$$f \colon A \longrightarrow A'$$

in $\mathscr{A}$. We define $\nabla(B)(f)$ by case distinction.

[7] If we consider for $\mathscr{A}$ and $\mathscr{B}$ the categories $\mathcal{O}(X)$ and **Set** respectively, then $\nabla(B)(X) = B$, and $\nabla(B)(U) = \{*\}$ for every proper open subset U of X.

Case 1. If A is terminal in $\mathscr{A}$, then it follows from the existence of the morphism f that the object A' is again terminal in $\mathscr{A}$ (by assumption on $\mathscr{A}$) whence we can choose the morphism $\nabla(B)(f)$ as 1_B.

Case 2. If A is not terminal in $\mathscr{A}$, then we have $\nabla(B)(A) = T(\mathscr{B})$. We then let $\nabla(B)(f)$ be the unique morphism from $\nabla(B)(A')$ to $T(\mathscr{B})$.

Let us show that this assignment $\nabla(B)$ is a contravariant functor from $\mathscr{A}$ to $\mathscr{B}$. For this, we need to check that $\nabla(B)$ is compatible with identities and with composition of morphisms.

- Let A be on object of $\mathscr{A}$. To compute the action of $\nabla(B)$ on the morphism 1_A we have two cases to consider.

  Case 1. If A is terminal in $\mathscr{A}$, then $\nabla(B)(A)$ is defined as B and $\nabla(B)(1_A)$ is defined as 1_B, whence $\nabla(B)(1_A) = 1_{\nabla(B)(A)}$.

  Case 2. If A is not terminal in $\mathscr{A}$, then the morphisms $\nabla(B)(1_A)$ is defined as the unique morphism from $\nabla(B)(A)$ to $T(\mathscr{B})$. But $\nabla(B)(A)$ is defined as $T(\mathscr{B})$, so that $\nabla(B)(1_A)$ is the unique morphism from $T(\mathscr{B})$ to $T(\mathscr{B})$. This morphism is precisely $1_{T(\mathscr{B})}$, and thus $1_{\nabla(B)(A)}$.

  We have in either case that $\nabla(B)(1_A) = 1_{\nabla(B)(A)}$.

- Let
$$f : A \longrightarrow A' , \quad f' : A' \longrightarrow A''$$
  be two composable morphisms in $\mathscr{A}$. To compute the action of $\nabla(B)$ on the composite $f' \circ f$ we have two cases to consider.

  Case 1. Suppose that A is terminal in $\mathscr{A}$. It then follows from the existence of the morphisms f and f' that the objects A' and A'' are again terminal in $\mathscr{A}$ (by assumption on the category $\mathscr{A}$). It then further follows that the morphisms $\nabla(B)(f)$, $\nabla(B)(f')$ and $\nabla(B)(f' \circ f)$ are all three given by 1_B. Therefore,
$$\nabla(B)(f) \circ \nabla(B)(f') = 1_B \circ 1_B = 1_B = \nabla(B)(f' \circ f) .$$

  Case 2. Suppose that the object A is not terminal in $\mathscr{A}$. The object $\nabla(B)(A)$ is then given by the terminal object $T(\mathscr{B})$. It follows that the two morphisms $\nabla(B)(f' \circ f)$ and $\nabla(B)(f) \circ \nabla(B)(f')$ are equal, since they both go from $\nabla(B)(A'')$ to $T(\mathscr{B})$.

  We have in either case that $\nabla(B)(f' \circ f) = \nabla(B)(f) \circ \nabla(B)(f')$.

With this, we have shown that the assignment $\nabla(B)$ is indeed a contravariant functor from $\mathscr{A}$ to $\mathscr{B}$.

We will now explain how the functor $\nabla(B)$ depends itself functorially on B. For this, let

$$g\colon\ B \longrightarrow B'$$

be a morphism in $\mathscr{B}$. We define a transformation $\nabla(g) = (\nabla(g)_A)_{A \in \mathrm{Ob}(\mathscr{A})}$ from $\nabla(B)$ to $\nabla(B')$ via the components

$$\nabla(g)_A := \begin{cases} g & \text{if } A \text{ is terminal in } \mathscr{A}, \\ 1_{T(\mathscr{B})} & \text{otherwise.} \end{cases}$$

This transformation is natural. To see this, we consider a morphism

$$f\colon\ A \longrightarrow A'$$

in $\mathscr{A}$, and we need to show that the square diagram

$$\begin{tikzcd} \nabla(B)(A') \arrow[r, "\nabla(B)(f)"] \arrow[d, "\nabla(g)_{A'}"'] & \nabla(B)(A) \arrow[d, "\nabla(g)_A"] \\ \nabla(B')(A') \arrow[r, "\nabla(B')(f)"] & \nabla(B')(A) \end{tikzcd} \tag{2.3}$$

commutes. We have two cases to consider.

Case 1. Suppose that the object A is terminal in $\mathscr{A}$. It then follows from the existence of the morphism f that the object A' is also terminal in $\mathscr{A}$ (by assumption on the category $\mathscr{A}$). The square diagram (2.3) can then be simplified as follows:

$$\begin{tikzcd} B \arrow[r, "1_B"] \arrow[d, "g"'] & B \arrow[d, "g"] \\ B' \arrow[r, "1_{B'}"] & B' \end{tikzcd}$$

This diagram commutes.

Case 2. Suppose that the object A is not terminal in $\mathscr{A}$. The object $\nabla(B')(A)$ is then given by the terminal object $T(\mathscr{B})$. It follows that the diagram (2.3) commutes because there exists precisely one morphism from $\nabla(B)(A')$ to $T(\mathscr{B})$.

We find in both cases that the diagram (2.3) commutes. This shows that $\nabla(g)$ is indeed a natural transformation from $\nabla(B)$ to $\nabla(B')$.

Let us now show that the assignment ∇ is functorial. To this end, we need to check that ∇ is compatible with identity morphisms and with composition of morphisms.

- For every object B of $\mathscr{B}$ we have the equalities

$$\begin{aligned} \nabla(1_B)_A &= \begin{cases} 1_B & \text{if } A \text{ is terminal in } \mathscr{A}, \\ 1_{T(\mathscr{B})} & \text{otherwise}, \end{cases} \\ &= \begin{cases} 1_{\nabla(B)(A)} & \text{if } A \text{ is terminal in } \mathscr{A}, \\ 1_{\nabla(B)(A)} & \text{otherwise}, \end{cases} \\ &= 1_{\nabla(B)(A)} \\ &= (1_{\nabla(B)})_A \end{aligned}$$

  for every object A of $\mathscr{A}$, and therefore the equality

$$\nabla(1_B) = 1_{\nabla(B)} .$$

- For every two composable morphisms

$$g : B \longrightarrow B' , \quad g' : B' \longrightarrow B''$$

  in $\mathscr{B}$ we have the equalities

$$\begin{aligned} &(\nabla(g') \circ \nabla(g))_A \\ &= \nabla(g')_A \circ \nabla(g)_A \\ &= \left(\begin{cases} g' & \text{if } A \text{ is terminal in } \mathscr{A}, \\ 1_{T(\mathscr{B})} & \text{otherwise}, \end{cases} \right) \circ \left(\begin{cases} g & \text{if } A \text{ is terminal in } \mathscr{A}, \\ 1_{T(\mathscr{B})} & \text{otherwise}, \end{cases} \right) \\ &= \begin{cases} g' \circ g & \text{if } A \text{ is terminal in } \mathscr{A}, \\ 1_{T(\mathscr{B})} & \text{otherwise}, \end{cases} \\ &= \nabla(g' \circ g)_A \end{aligned}$$

  for every object A of $\mathscr{A}$, and therefore the equality

$$\nabla(g') \circ \nabla(g) = \nabla(g' \circ g) .$$

This shows the functoriality of ∇.

We will now show that the functor ∇ is right adjoint to the functor Γ. Let F be a contravariant functor from $\mathscr{A}$ to $\mathscr{B}$ and let B be an object of $\mathscr{B}$. A natural transformation β from F to $\nabla(B)$ has as its component at $T(\mathscr{A})$ a morphism from the object $F(T(\mathscr{A}))$ to the object $\nabla(B)(T(\mathscr{A}))$. These objects are given by $F(T(\mathscr{A})) = \Gamma(F)$ and $\nabla(B)(T(\mathscr{A})) = B$ respectively. We have therefore a well-defined map

$$\overline{(-)} : [\mathscr{A}^{\mathrm{op}}, \mathscr{B}](F, \nabla(B)) \longrightarrow \mathscr{B}(\Gamma(F), B), \quad \beta \longmapsto \beta_{T(\mathscr{A})}. \tag{2.4}$$

We will show in the following that this map is bijective, and natural in both F and in B.

To show that the map (2.4) is injective let β be a natural transformation from F to $\nabla(B)$. We need to show that β is uniquely determined by its component $\beta_{T(\mathscr{A})}$, which we shall denote by g. We show by case distinction that for every object A of $\mathscr{A}$, the morphism β_A from $F(A)$ to $\nabla(B)(A)$ is uniquely determined by g.

Case 1. Suppose the object A is non-terminal in $\mathscr{A}$. The object $\nabla(B)(A)$ is then given by the terminal object $T(\mathscr{B})$. There exists precisely one morphism from $F(A)$ to $T(\mathscr{B})$, whence β_A must be this morphism.

Case 2. Suppose that the object A is terminal in $\mathscr{A}$. The object $\nabla(B)(A)$ is then given by B. Both A and $T(\mathscr{A})$ are terminal objects in $\mathscr{A}$, whence the unique morphism h from $T(\mathscr{A})$ to A is an isomorphism. It follows from the naturality of β that the square diagram

$$\begin{tikzcd} F(A) \arrow[r, "\beta_A"] \arrow[d, "F(h)"'] & \nabla(B)(A) \arrow[d, "\nabla(B)(h)"] \\ F(T(\mathscr{A})) \arrow[r, "\beta_{T(\mathscr{A})}"] & \nabla(B)(T(\mathscr{A})) \end{tikzcd}$$

commutes. This diagram simplifies as follows:

$$\begin{tikzcd} F(A) \arrow[r, "\beta_A"] \arrow[d, "F(h)"'] & B \arrow[d, "1_B"] \\ F(T(\mathscr{A})) \arrow[r, "g"] & B \end{tikzcd}$$

We find that the morphism β_A is given by the composite $g \circ F(h)$. It is therefore uniquely determined by g.

We have thus shown that the map (2.4) is injective.

To show that is it surjective, let g be a morphism from $\Gamma(F)$ to B, i.e., a morphism from $F(T(\mathscr{A}))$ to B. We need to construct a natural transformation β from F to $\nabla(B)$ with $\beta_{T(\mathscr{A})} = g$. We define the components β_A via case distinction:

Case 1. If A is a non-terminal object of $\mathscr{A}$, then $\nabla(B)(A)$ is the terminal object $T(\mathscr{B})$. We then let β_A be the unique morphism from $F(A)$ to $\nabla(B)(A)$.

Case 2. If A is terminal in $\mathscr{A}$, then there exists a unique morphism h from the object $T(\mathscr{A})$ to A (and this morphism is an isomorphism since $T(\mathscr{A})$ is also terminal in $\mathscr{A}$). We then let β_A be the composite $g \circ F(h)$.

(We note that in the case of $A = T(\mathscr{A})$, we have $h = 1_{T(\mathscr{A})}$, therefore $F(h) = 1_{F(T(\mathscr{A}))}$, and thus $\beta_{T(\mathscr{A})} = g \circ F(h) = g$.)

To prove the naturality of β let

$$f : A \longrightarrow A'$$

be an arbitrary morphism in $\mathscr{A}$. We need to show that the square diagram

$$\begin{tikzcd} F(A') \arrow[r, "F(f)"] \arrow[d, "\beta_{A'}"'] & F(A) \arrow[d, "\beta_A"] \\ \nabla(B)(A') \arrow[r, "\nabla(B)(f)"] & \nabla(B)(A) \end{tikzcd}$$

commutes. We do so by case distinction.

Case 1. Suppose that the object A is non-terminal in $\mathscr{A}$. The object $\nabla(B)(A)$ is then given by the terminal object $T(\mathscr{B})$. It follows that the above square diagram commutes because there exists precisely one morphism from $F(A')$ to $T(\mathscr{B})$.

Case 2. Suppose that the object A is terminal in $\mathscr{A}$. It then follows from the existence of the morphism f that the object A' is again terminal in A

(by assumption on $\mathscr{A}$). There exist unique morphisms h and h' of the forms

$$h : T(\mathscr{A}) \longrightarrow A\,, \quad h' : T(\mathscr{A}) \longrightarrow A'\,.$$

The two morphisms h and h' are isomorphisms and fit into the following commutative diagram:

$$\begin{tikzcd}
 & T(\mathscr{A}) \arrow[dl, "h'"'] \arrow[dr, "h"] & \\
A' & & A \arrow[ll, "f"']
\end{tikzcd} \tag{2.5}$$

We can now consider the following diagram:

$$\begin{tikzcd}
 & F(T(\mathscr{A})) \arrow[dd, "g"] & \\
F(A') \arrow[ur, "F(h')"] \arrow[rr, "F(f)"] \arrow[dd, "\beta_{A'}"'] & & F(A) \arrow[ul, "F(h)"'] \arrow[dd, "\beta_A"] \\
 & \nabla(B)(T(\mathscr{A})) & \\
\nabla(B)(A') \arrow[ur, "\nabla(B)(h')"] \arrow[rr, "\nabla(B)(f)"] & & \nabla(B)(A) \arrow[ul, "\nabla(B)(h)"']
\end{tikzcd} \tag{2.6}$$

The two triangular sides of this diagram commute because the diagram (2.5) commutes. The background part of this diagram is given by

$$\begin{tikzcd}
F(A') \arrow[r, "F(h')"] \arrow[d, "\beta_{A'}"'] & F(T(\mathscr{A})) \arrow[d, "g"] & F(A) \arrow[l, "F(h)"'] \arrow[d, "\beta_A"] \\
\nabla(B)(A') \arrow[r, "\nabla(B)(h')"] & \nabla(B)(T(\mathscr{A})) & \nabla(B)(A) \arrow[l, "\nabla(B)(h)"']
\end{tikzcd}$$

and can be simplified as follows:

$$\begin{tikzcd} F(A') \arrow[r, "F(h')"] \arrow[d, "g \circ F(h')"'] & F(T(\mathscr{A})) \arrow[d, "g"] & F(A) \arrow[l, "F(h)"'] \arrow[d, "g \circ F(h)"] \\ B \arrow[r, "1_B"] & B & B \arrow[l, "1_B"'] \end{tikzcd}$$

This simplified diagram commutes. We have thus seen that all non-frontal sides of the diagram (2.6) commute. It follows that its front side also commutes because

$$\begin{aligned} \nabla(B)(h) \circ \beta_A \circ F(f) &= g \circ F(h) \circ F(f) \\ &= g \circ F(h') \\ &= \nabla(B)(h') \circ \beta_{A'} \\ &= \nabla(B)(h) \circ \nabla(B)(f) \circ \beta_{A'} \end{aligned}$$

with $\nabla(B)(h) = 1_B$ being an isomorphism.

We have thus proven the naturality of the transformation β, which in turn shows the surjectivity of the map (2.4).

To show that the map (2.4) is natural in F, we consider a natural transformation

$$\alpha : F \Longrightarrow F'$$

between two contravariant functors F and F' from $\mathscr{A}$ to $\mathscr{B}$. We need to show that the diagram

$$\begin{tikzcd} {[\mathscr{A}^{\mathrm{op}}, \mathscr{B}](F, \nabla(B))} \arrow[r, "\overline{(-)}"] & \mathscr{B}(\Gamma(F), B) \\ {[\mathscr{A}^{\mathrm{op}}, \mathscr{B}](F', \nabla(B))} \arrow[r, "\overline{(-)}"] \arrow[u, "\alpha^*"] & \mathscr{B}(\Gamma(F'), B) \arrow[u, "\Gamma(\alpha)^*"'] \end{tikzcd}$$

commutes. For this, we observe for every element β of the bottom-left corner of this diagram the equalities

$$\overline{\alpha^*(\beta)} = \overline{\beta \circ \alpha} = (\beta \circ \alpha)_{T(\mathscr{A})} = \beta_{T(\mathscr{A})} \circ \alpha_{T(\mathscr{A})} = (\alpha_{T(\mathscr{A})})^*(\beta_{T(\mathscr{A})}) = \Gamma(\alpha)^*(\overline{\beta}) .$$

To show that the map (2.4) is natural in B, we consider an arbitrary morphism

$$g : B \longrightarrow B'$$

in $\mathscr{B}$. We need to show that the diagram

$$\begin{tikzcd}
{[\mathscr{A}^{\mathrm{op}}, \mathscr{B}](F, \nabla(B))} \arrow[r, "\overline{(-)}"] \arrow[d, "\nabla(g)_*"'] & \mathscr{B}(\Gamma(F), B) \arrow[d, "g_*"] \\
{[\mathscr{A}^{\mathrm{op}}, \mathscr{B}](F, \nabla(B'))} \arrow[r, "\overline{(-)}"] & \mathscr{B}(\Gamma(F), B')
\end{tikzcd}$$

commutes. For this, we observe for every element α of the top-left corner of this diagram the equalities

$$g_*(\overline{\alpha}) = g \circ \overline{\alpha} = \nabla(g)_{T(\mathscr{A})} \circ \alpha_{T(\mathscr{A})} = (\nabla(g) \circ \alpha)_{T(\mathscr{A})} = \overline{\nabla(g) \circ \alpha} = \overline{\nabla(g)_*(\alpha)}\,.$$

We have altogether constructed a functor ∇ from $\mathscr{B}$ to $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}]$ that is right adjoint to Γ.

The functors Π and Λ

We have now seen that for suitable categories $\mathscr{A}$ and $\mathscr{B}$, the diagonal functor

$$\Delta_{\mathscr{A},\mathscr{B}} : \mathscr{B} \longrightarrow [\mathscr{A}^{\mathrm{op}}, \mathscr{B}]$$

admits a right adjoint

$$\Gamma_{\mathscr{A},\mathscr{B}} : [\mathscr{A}^{\mathrm{op}}, \mathscr{B}] \longrightarrow \mathscr{B}\,,$$

which in turn admits a right adjoint

$$\nabla_{\mathscr{A},\mathscr{B}} : \mathscr{B} \longrightarrow [\mathscr{A}^{\mathrm{op}}, \mathscr{B}]\,.$$

To construct the functors Π and Λ we make the following observation.

Proposition 2.A. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories and let

$$F : \mathscr{A} \longrightarrow \mathscr{B}\,, \quad G : \mathscr{B} \longrightarrow \mathscr{A}$$

be two functors. We may regard F and G as functors

$$F' : \mathscr{A}^{\mathrm{op}} \longrightarrow \mathscr{B}^{\mathrm{op}}\,, \quad G' : \mathscr{B}^{\mathrm{op}} \longrightarrow \mathscr{A}^{\mathrm{op}}\,.$$

Then, F is left adjoint to G if and only if F' is right adjoint to G'.

Proof. Suppose that F is left adjoint to G. This means that we have a bijection

$$\Phi_{A,B} : \mathscr{B}(F(A), B) \longrightarrow \mathscr{A}(A, G(B))$$

for every object A of $\mathscr{A}$ and every object B of $\mathscr{B}$, such that these bijections are "natural" in the following sense: for all morphisms

$$f : A \longrightarrow A', \quad g : B \longrightarrow B'$$

in $\mathscr{A}$ and $\mathscr{B}$ respectively, the following diagram commutes:

$$\begin{tikzcd}
\mathscr{B}(F(A'), B) \arrow[r, "\Phi_{A',B}"] \arrow[d, "g\circ(-)\circ F(f)"'] & \mathscr{A}(A', G(B)) \arrow[d, "G(g)\circ(-)\circ f"] \\
\mathscr{B}(F(A), B') \arrow[r, "\Phi_{A,B'}"] & \mathscr{B}(A, G(B'))
\end{tikzcd}$$

We may regard the bijections $\Phi_{A,B}$ as bijections

$$\Phi'_{B^{\mathrm{op}},A^{\mathrm{op}}} : \mathscr{B}^{\mathrm{op}}(B^{\mathrm{op}}, F'(A^{\mathrm{op}})) \longrightarrow \mathscr{A}^{\mathrm{op}}(G'(B^{\mathrm{op}}), A^{\mathrm{op}}) .$$

The above commutative diagram can be rewritten as follows:

$$\begin{tikzcd}
\mathscr{B}^{\mathrm{op}}(B^{\mathrm{op}}, F'((A')^{\mathrm{op}})) \arrow[r, "\Phi'_{B^{\mathrm{op}},(A')^{\mathrm{op}}}"] \arrow[d, "F'(f^{\mathrm{op}})\circ(-)\circ g^{\mathrm{op}}"'] & \mathscr{A}^{\mathrm{op}}(G'(B^{\mathrm{op}}), (A')^{\mathrm{op}}) \arrow[d, "f^{\mathrm{op}}\circ(-)\circ G'(g^{\mathrm{op}})"] \\
\mathscr{B}^{\mathrm{op}}((B')^{\mathrm{op}}, F'(A^{\mathrm{op}})) \arrow[r, "\Phi_{(B')^{\mathrm{op}},A^{\mathrm{op}}}"] & \mathscr{B}^{\mathrm{op}}(G'((B')^{\mathrm{op}}), A^{\mathrm{op}})
\end{tikzcd}$$

The commutativity of this diagram tells us that the bijections $\Phi_{B^{\mathrm{op}},A^{\mathrm{op}}}$ are again natural. The existence of such natural bijections shows that the functor F' is right adjoint to the functor G'.

Suppose conversely that F' is right adjoint to G'. This means that G' is left adjoint to F'. It follows for the functors

$$F'' : \mathscr{A}^{\mathrm{opop}} \longrightarrow \mathscr{B}^{\mathrm{opop}}, \quad G'' : \mathscr{B}^{\mathrm{opop}} \longrightarrow \mathscr{A}^{\mathrm{opop}}$$

that F'' is left adjoint to G''. Under the equalities of $\mathscr{A}^{\mathrm{opop}}$ and $\mathscr{B}^{\mathrm{opop}}$ with $\mathscr{A}$ and $\mathscr{B}$ respectively, the functors F'' and G'' correspond to the functors F and G respectively. Therefore, F is left adjoint to G. ∎

We have the chain of adjunctions

$$\Delta_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}} \dashv \Gamma_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}} \dashv \nabla_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}}$$

between the two categories $\mathscr{B}^{\mathrm{op}}$ and $[\mathscr{A}^{\mathrm{opop}}, \mathscr{B}^{\mathrm{op}}]$. Under the isomorphism

$$[\mathscr{A}^{\mathrm{opop}}, \mathscr{B}^{\mathrm{op}}] \cong [\mathscr{A}^{\mathrm{op}}, \mathscr{B}]^{\mathrm{op}}$$

from Exercise 1.3.27, we have thus a chain of adjunctions between the two categories $\mathscr{B}^{\mathrm{op}}$ and $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}]^{\mathrm{op}}$. According to Proposition 2.A, we get an induced chain of adjunctions

$$\nabla'_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}} \dashv \Gamma'_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}} \dashv \Delta'_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}}$$

between $\mathscr{B}$ and $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}]$. Let us derive explicit descriptions of these three functors:

- Let us abbreviate the functor $\Delta_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}}$ by Δ.

  Let B be an object of $\mathscr{B}$. The functor Δ assigns to the object B^{op} constant functor at B^{op}. The functor Δ' therefore assigns to the object B^{opop} the constant functor at B^{opop}. Equivalently, it assigns to the object B the constant functor at B.

  Let $g : B \to B'$ be a morphism in $\mathscr{B}$. The functor Δ assigns to the morphism g^{op} the natural transformation $\Delta(g^{\mathrm{op}})$ from the contravariant functor $\Delta(B^{\mathrm{op}})$ to the contravariant functor $\Delta((B')^{\mathrm{op}})$ whose components are given by

  $$\Delta(g^{\mathrm{op}})_{A^{\mathrm{op}}} = g^{\mathrm{op}}$$

  for every object A of $\mathscr{A}$.[8] The functor Δ' therefore assigns to the morphism g^{opop} the natural transformation $\Delta'(g^{\mathrm{opop}})$ whose components are gives by

  $$\Delta(g^{\mathrm{opop}})_{A^{\mathrm{opop}}} = g^{\mathrm{opop}}$$

  for every object A of $\mathscr{A}$. Equivalently,

  $$\Delta(g)_A = g$$

  for every object A of $\mathscr{A}$.

  We find from these above descriptions that the functor $\Delta' = \Delta'_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}}$ coincides with the diagonal functor $\Delta_{\mathscr{A},\mathscr{B}}$.

[8] The functors $\Delta(B^{\mathrm{op}})$ and $\Delta((B')^{\mathrm{op}})$ are objects of the functor category $[\mathscr{A}^{\mathrm{opop}}, \mathscr{B}^{\mathrm{op}}]$. We view this functor category as the category of contravariant functors from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}^{\mathrm{op}}$. We are therefore indexing the components of $\Delta(g^{\mathrm{op}})$ by the objects of $\mathscr{A}^{\mathrm{op}}$, and not by the objects of $\mathscr{A}^{\mathrm{opop}}$.

- Let us abbreviate the functor $\Gamma_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}}$ as Γ.

  The functor Γ, going from the category $[\mathscr{A}^{\mathrm{opop}}, \mathscr{B}^{\mathrm{op}}]$ to the category $\mathscr{B}^{\mathrm{op}}$, is the evaluation functor at $T(\mathscr{A}^{\mathrm{op}})$ if we regard $[\mathscr{A}^{\mathrm{opop}}, \mathscr{B}^{\mathrm{op}}]$ as the category of contravariant functors from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}^{\mathrm{op}}$. As a functor from $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}]^{\mathrm{op}}$ to $\mathscr{B}^{\mathrm{op}}$, Γ is thus given by evaluation at $T(\mathscr{A}^{\mathrm{op}})$.

  The functor Γ' is thus again given by evaluation at $T(\mathscr{A}^{\mathrm{op}})$ if we view the category $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}]$ as the category of covariant functors from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}$. If we view $[\mathscr{A}^{\mathrm{op}}, \mathscr{B}]$ as the category of contravariant functors from $\mathscr{A}$ to $\mathscr{B}$ instead, then Γ' is therefore given by evaluation at $I(\mathscr{A})$.

- Let us abbreviate the functor $\nabla_{\mathscr{A}^{\mathrm{op}},\mathscr{B}^{\mathrm{op}}}$ by ∇. This functor goes from the category $\mathscr{B}^{\mathrm{op}}$ to the functor category $[\mathscr{A}^{\mathrm{opop}}, \mathscr{B}^{\mathrm{op}}]$. We view this functor category as the category of contravariant functors from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}^{\mathrm{op}}$.

  Let B be an object of $\mathscr{B}$. The contravariant functor $\nabla(B^{\mathrm{op}})$ from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}^{\mathrm{op}}$ is given on objects by
  $$\nabla(B^{\mathrm{op}})(A^{\mathrm{op}}) = \begin{cases} B^{\mathrm{op}} & \text{if } A^{\mathrm{op}} \text{ is terminal in } \mathscr{A}^{\mathrm{op}}, \\ T(\mathscr{B}^{\mathrm{op}}) & \text{otherwise,} \end{cases}$$
  for every object A of $\mathscr{A}$. When we regard $\nabla(B^{\mathrm{op}})$ as a contravariant functor from $\mathscr{A}$ to $\mathscr{B}$, then it is given by
  $$\nabla(B^{\mathrm{op}})(A) = \begin{cases} B & \text{if } A \text{ is initial in } \mathscr{A}, \\ I(\mathscr{B}) & \text{otherwise,} \end{cases}$$
  for every object A of $\mathscr{A}$. The functor $\nabla'(B)$ from $\mathscr{A}$ to $\mathscr{B}$ is therefore given by
  $$\nabla'(B)(A) = \begin{cases} B & \text{if } A \text{ is initial in } \mathscr{A}, \\ I(\mathscr{B}) & \text{otherwise,} \end{cases}$$
  for every object A of $\mathscr{A}$.

  Let $g \colon B \to B'$ be a morphism in $\mathscr{B}$. The natural transformation $\nabla(g^{\mathrm{op}})$ from the functor $\nabla((B')^{\mathrm{op}})$ to the functor $\nabla(B^{\mathrm{op}})$ is given by the components
  $$\nabla(g^{\mathrm{op}})_{A^{\mathrm{op}}} = \begin{cases} g^{\mathrm{op}} & \text{if } A^{\mathrm{op}} \text{ is terminal in } \mathscr{A}^{\mathrm{op}}, \\ 1_{T(\mathscr{B}^{\mathrm{op}})} & \text{otherwise,} \end{cases}$$
  for every object A of $\mathscr{A}$. If we regard $\nabla((B')^{\mathrm{op}})$ and $\nabla(B^{\mathrm{op}})$ as functors from $\mathscr{A}$ to $\mathscr{B}$ instead, then $\nabla(g^{\mathrm{op}})$ corresponds to the natural transformation α from $\nabla(B^{\mathrm{op}})$ to $\nabla((B')^{\mathrm{op}})$ with components
  $$\alpha_A = \begin{cases} g & \text{if } A \text{ is initial in } \mathscr{A}, \\ 1_{I(\mathscr{B})} & \text{otherwise,} \end{cases}$$

for every object A of $\mathscr{A}$. The natural transformation $\nabla'(g)$ from $\nabla'(B)$ to $\nabla'(B')$ is therefore given by the components

$$\nabla'(g)_A = \begin{cases} g & \text{if } A \text{ is initial in } \mathscr{A}, \\ 1_{I(\mathscr{B})} & \text{otherwise,} \end{cases}$$

for every object A of $\mathscr{A}$.

We have thus derived explicit constructions for the functors Λ and Π:

- The functor Π is the evaluation functor at the initial object of $\mathscr{A}$.[9]
- Let B be an object of $\mathscr{B}$. The contravariant functor $\Lambda(B)$ from $\mathscr{A}$ to $\mathscr{B}$ is given by on objects by

  $$\Lambda(B)(A) = \begin{cases} B & \text{if } A \text{ is initial in } \mathscr{A}, \\ I(\mathscr{B}) & \text{otherwise,} \end{cases}$$

  for every object A of $\mathscr{A}$,[10] and on morphisms by

  $$\Lambda(B)(f) = \begin{cases} 1_B & \text{if } A \text{ is initial in } \mathscr{A}, \\ 1_{I(\mathscr{B})} & \text{otherwise,} \end{cases}$$

  for every morphism f in $\mathscr{A}$. For every morphism $g : B \to B'$ in $\mathscr{B}$, the natural transformation $\Lambda(g)$ from $\Lambda(B)$ to $\Lambda(B')$ is given by the components

  $$\Lambda(g)_A = \begin{cases} g & \text{if } A \text{ is initial in } \mathscr{A}, \\ 1_{I(\mathscr{B})} & \text{otherwise,} \end{cases}$$

  for every object A of $\mathscr{A}$.

2.2 Adjunctions via units and counits

Exercise 2.2.10

Suppose first that condition (a) is satisfied. The two relations $a \leq g(f(a))$ and $f(g(b)) \leq b$ from condition (b) are then equivalent to the true relations $f(a) \leq f(a)$ and $g(b) \leq g(b)$ respectively. We hence find that condition (b) follows from condition (a).

---

[9] If we consider for $\mathscr{A}$ and $\mathscr{B}$ the categories $\mathcal{O}(X)$ and **Set** respectively, then Π is given by the evaluation functor at $\emptyset$.

[10] If we choose for $\mathscr{A}$ and $\mathscr{B}$ the categories $\mathcal{O}(X)$ and **Set** respectively, then the functor $\Lambda(B)$ is given on objects by $\Lambda(B)(\emptyset) = B$ and $\Lambda(B)(U) = \emptyset$ for every non-empty open subset U of X.

If on the other hand condition (b) is satisfied, then it follows that

$$f(a) \leq b \implies g(f(a)) \leq g(b) \implies a \leq g(b)$$

and

$$a \leq g(b) \implies f(a) \leq f(g(b)) \implies f(a) \leq b$$

for any two elements a and b of A and B respectively. We hence find that condition (a) follows from condition (b).

Exercise 2.2.11

(a)

We denote the subcategories $\mathbf{Fix}(GF)$ and $\mathbf{Fix}(FG)$ of $\mathscr{A}$ and $\mathscr{B}$ by $\mathscr{A}'$ and $\mathscr{B}'$ respectively.

We start off by showing that the functor F restrict to a functor from $\mathscr{A}'$ to $\mathscr{B}'$. Let A be an object of $\mathscr{A}'$. The morphisms η_A from A to $GF(A)$ is an isomorphism, whence its image under F is again an isomorphism, this time from $F(A)$ to $FGF(A)$. We know from the triangle identities that

$$1_{F(A)} = \varepsilon_{F(A)} \circ F(\eta_A) .$$

It follows that the morphism $\varepsilon_{F(A)}$ is an isomorphism as both $1_{F(A)}$ and $F(\eta_A)$ are isomorphisms. This tells us that object $F(A)$ is contained in $\mathscr{B}'$. We therefore find that the functor F restricts to a functor F' from $\mathscr{A}'$ to $\mathscr{B}'$. (We don't need to worry about the action of F on morphisms in $\mathscr{A}'$ because $\mathscr{B}'$ is a full subcategory of $\mathscr{B}$.)

We find in the same way (by using the other triangle identity) that the functor G restricts to functor G' from $\mathscr{B}'$ to $\mathscr{A}'$.

The natural transformation η from $1_{\mathscr{A}}$ to GF restricts to a natural transformation from $1_{\mathscr{A}'}$ to $G'F'$, and the natural transformation ε from FG to $1_{\mathscr{B}}$ restricts to a natural transformation ε' from $F'G'$ to $1_{\mathscr{B}'}$. We observe that the subcategories $\mathscr{A}'$ and $\mathscr{B}'$ of $\mathscr{A}$ and $\mathscr{B}$ are chosen precisely in such a way that all components of η' and ε' are isomorphisms. The natural transformations η' and ε' are therefore natural isomorphisms.

We have now overall the two categories $\mathscr{A}'$ and $\mathscr{B}'$, the two functors

$$F' : \mathscr{A}' \longrightarrow \mathscr{B}' , \quad G' : \mathscr{B}' \longrightarrow \mathscr{A}' ,$$

and the two natural isomorphisms

$$\eta' : 1_{\mathscr{A}'} \Longrightarrow G'F' , \quad \varepsilon' : F'G' \Longrightarrow 1_{\mathscr{B}'} .$$

We have, in other words, an equivalence of categories $(F', G', \eta', \varepsilon')$ between the two categories $\mathscr{A}'$ and $\mathscr{B}'$.

(b)

Example 2.1.3, (a) Given any set S, the unit map η_S from S to $UF(S)$ is not surjective because $UF(S)$ contains the zero vector, which does not lie in the image of η_S. We therefore find that the subcategory $\mathbf{Fix}(UF)$ of $\mathbf{Set}$ is empty.

It follows that the category $\mathbf{Fix}(FU)$ is also empty, since it is equivalent to $\mathbf{Fix}(UF)$.

Example 2.1.3, (b) In the same way as in the previous example, we find that both $\mathbf{Fix}(UF)$ and $\mathbf{Fix}(FU)$ are empty.

Example 2.1.3, (c) The composite functor UF assigns to each group G its abelianization G^{ab}. The unit morphism η_G is the canonical homomorphism of groups from G to G^{ab}. This homomorphism is an isomorphism if and only if the group G is abelian. We hence find that the $\mathbf{Fix}(UF)$ is $\mathbf{Ab}$, i.e., the full subcategory of $\mathbf{Grp}$ whose objects are abelian groups.

We find similarly that the subcategory $\mathbf{Fix}(FU)$ of $\mathbf{Ab}$ is all of $\mathbf{Ab}$.

The functor U restricts to the identity functor of $\mathbf{Ab}$, while the restriction of the functor F to an endofunctor of $\mathbf{Ab}$ assigns to each abelian group A its abelianization A^{ab}.

Example 2.1.3, (d), the functors F and U The counit morphism ε_G is an isomorphism for every group G, whence the category $\mathbf{Fix}(FU)$ is all of $\mathbf{Grp}$.

Given a monoid M, the unit morphism η_M from M to $UF(M)$ can only be an isomorphism if M was already a group to begin with. Conversely, if M is a group, then η_M will be an isomorphism. We hence find that the subcategory $\mathbf{Fix}(UF)$ of $\mathbf{Mon}$ is $\mathbf{Grp}$.

The functor U restricts to the identity functor of $\mathbf{Grp}$, whereas the restriction of the functor F will typically change the underlying set of a group, and is therefore not the identity functor.

Example 2.1.3, (d), the functors U and R The composite RU as the identity functor, and the unit η is the identity natural transformation. The category $\mathbf{Fix}(RU)$ is therefore all of $\mathbf{Grp}$.

The composite UR assigns to each monoid its submonoid $M^\times$ of units, and the counit morphism ε_M is for every monoid M given by the inclusion from $M^\times$ into M; this inclusion is an isomorphism if and only if every element of M is invertible, i.e., if and only if the monoid M was already a group to begin with. The subcategory $\mathbf{Fix}(UR)$ of $\mathbf{Mon}$ is therefore given by $\mathbf{Grp}$.

Both U and R restrict to the identity functors of $\mathbf{Grp}$, and the restrictions of η and ε are the identity natural transformation of this identity functor.

Example 2.1.5, the functors D and U The composite DU assigns to each topological space X the discrete topological space with the same underlying set as X. The unit morphism η_X is the identity map from this discrete space to X. It is an isomorphism if and only if the space X is discrete, whence the full subcategory $\mathbf{Fix}(DU)$ of $\mathbf{Top}$ consists precisely of the discrete topological spaces.

The composite UD is the identity functor, and the counit ε of the adjunction is the identity natural transformation. The subcategory $\mathbf{Fix}(UD)$ of $\mathbf{Set}$ is therefore all of $\mathbf{Set}$.

The restriction of η is the identity natural transformation of the identity functor of the category of discrete topological spaces.

Example 2.1.5, the functors U and I The composite IU assigns to each topological space X the indiscrete topological space with the same underlying set as X, and the counit morphism ε_X is the identity map from X to this indiscrete space. This map is an isomorphism if and only if the space X is indiscrete, whence the full subcategory $\mathbf{Fix}(IU)$ of $\mathbf{Top}$ consists precisely of the indiscrete topological spaces.

The composite UI is the identity functor, and the unit η of the adjunction is the identity natural transformation. The subcategory $\mathbf{Fix}(UI)$ of $\mathbf{Set}$ is therefore all of $\mathbf{Set}$.

The restriction of ε is the identity natural transformation of the identity functor of the category of indiscrete topological spaces.

Example 2.1.6 We denote the given functors by

$$F := (-) \times B\,, \quad G := (-)^B\,.$$

The composite GF is given on objects by

$$GF(A) = (A \times B)^B$$

for every set A, and the unit natural transformation η is given by

$$\eta_A \colon A \longrightarrow (A \times B)^B\,, \quad a \longmapsto [b \longmapsto (a, b)]$$

for every set A. The composite FG, on the other hand, is given on objects by

$$FG(A) = A^B \times B\,,$$

for every set A, and the counit natural transformation ε is given by the evaluation map

$$\varepsilon_A \colon A^B \times A \longrightarrow A\,, \quad (f, b) \longmapsto f(b)$$

for every set A. We distinguish in the following between three cases.

Case 1. Suppose that the set B is empty.

The set $GF(A)$ is then a singleton for every set A, whence the map η_A is an isomorphism if and only if the set A is a singleton. The full subcategory $\mathbf{Fix}(GF)$ of $\mathbf{Set}$ consists therefore of all the singleton sets. The restriction η' of the unit η has for every singleton set A as its component η_A the unique map from the singleton set A to the singleton set $(A \times B)^B$.

The set $FG(C)$ is empty, whence the counit map ε_C is an isomorphism if and only if the set C is empty. The subcategory $\mathbf{Fix}(FG)$ of $\mathbf{Set}$ therefore consists of a single object, namely the empty set.

The composite FG restricts to the identity functor on $\mathbf{Fix}(FG)$, and the counit ε to the identity natural transformation of this identity functor.

Case 2. Suppose that the set B is a singleton. Both η and ε are natural isomorphisms, whence both $\mathbf{Fix}(GF)$ and $\mathbf{Fix}(FG)$ are all of $\mathbf{Set}$.

Case 3. Suppose that the set B consists of at least two distinct elements.

The unit map η_A is an isomorphism if and only if the set A is empty. The subcategory $\mathbf{Fix}(GF)$ of $\mathbf{Set}$ therefore consists of only a single object, namely the empty set.

The counit map ε_C is an isomorphism if and only if the set C is empty. The subcategory $\mathbf{Fix}(FG)$ of $\mathbf{Set}$ therefore consists of only a single object, namely the empty set.

The restrictions of F and G are the identity functors on the full subcategory of $\mathbf{Set}$ whose single object is the empty set, and both η and ε restrict to the identity natural transformation of this identity functor.

Exercise 2.2.12

(a)

It follows from the first formula in Lemma 2.2.4 and the naturality of the unit η that

$$\overline{F(f)} = GF(f) \circ \eta_A = \eta_{A'} \circ f$$

for every morphism $f : A \to A'$ in $\mathscr{A}$. We can therefore express the action of the functor F on such a morphism f as

$$F(f) = \overline{\eta_{A'} \circ f}\,.$$

It follows that the functor F is faithful or full if and only if for every two objects A and A' of $\mathscr{A}$ the map

$$(\eta_{A'})_* : \mathscr{A}(A, A') \longrightarrow \mathscr{A}(A, GF(A'))$$

is injective, respectively surjective. By putting both of these observations together we find that F is full and faithful if and only if the above map $(\eta_{A'})_*$ is bijective for any two objects A and A' of $\mathscr{A}$. This is equivalent to each morphism $\eta_{A'}$ being an isomorphism by the upcoming Lemma 2.B, and therefore equivalent to η being a natural isomorphism.

Lemma 2.B. Let $\mathscr{A}$ be a category and let f be a morphism in $\mathscr{A}$ of the form

$$f : A \longrightarrow A'\,.$$

The following conditions on the morphism f are equivalent:

i. f is an isomorphism.

ii. The map $f_* : \mathscr{A}(A'', A) \to \mathscr{A}(A'', A')$ is bijective for every object A'' of $\mathscr{A}$.

iii. The map $f^* : \mathscr{A}(A', A'') \to \mathscr{A}(A, A'')$ is bijective for every object A'' of $\mathscr{A}$.

Proof. It suffices to prove the equivalence of the conditions i and ii. The equivalence of the conditions i and iii then follows by duality.

Suppose first that the morphism f is an isomorphism. The morphisms f and f^{-1} are mutually inverse, whence the two induced maps

$$f_* : \mathscr{A}(A'', A) \longrightarrow \mathscr{A}(A'', A'), \qquad (f^{-1})_* : \mathscr{A}(A'', A') \longrightarrow \mathscr{A}(A'', A)$$

are again mutually inverse. This shows that the map f_* is bijective. Thus, ii follows from i.

Suppose conversely that ii holds. By choosing A'' as A', we see that there exists a morphism g from A' to A with $1_{A'} = f_*(g)$. This means that

$$f \circ g = 1_{A'} .$$

We claim that also $g \circ f = 1_A$. To prove this, we note that both $g \circ f$ and 1_A are morphisms from A to A such that

$$f_*(g \circ f) = f \circ g \circ f = 1_{A'} \circ f = f = f \circ 1_A = f_*(1_A) .$$

It follows from the injectivity of f_* (for the case $A'' = A$) that indeed $g \circ f = 1_A$. ∎

We find dually the following: The functor G is faithful, respectively full, if and only if for every two objects B and B' of $\mathscr{B}$ the map

$$(\varepsilon_B)^* : \mathscr{B}(B, B') \longrightarrow \mathscr{B}(FG(B), B')$$

is injective, respectively surjective. Therefore, G is full and faithful if and only if the above map $(\varepsilon_B)^*$ is bijective for any two objects B and B' of $\mathscr{B}$. By Lemma 2.B this is equivalent to ε being a natural isomorphism.

Remark 2.C. We have actually shown the following stronger results for the left adjoint F, the right adjoint G, the unit η, and the counit ε.

1. F is faithful if and only if η is a monomorphism in each component.
2. G is faithful if and only if ε is an epimorphism in each component.
3. F is full if and only if η is a split epimorphism in each component.
4. G is full if and only if ε is a split monomorphism in each component.

These results can also be found in [Mac98, IV.3, Theorem 1].

(b)

Example 2.1.3, (a) The right adjoint functor U is not full. The given adjunction is therefore not a reflection.

Example 2.1.3, (b) The right adjoint functor U is not full. The given adjunction is therefore not a reflection.

Example 2.1.3, (c) The right adjoint functor U is full and faithful. The given adjunction is therefore a reflection.

Example 2.1.3, (d), the functors F and U The right adjoint functor U is full and faithful. The given adjunction is therefore a reflection.

Example 2.1.3, (d), the functors U and R The counit ε of the given adjunction has for every monoid M as its component ε_M the inclusion map from $M^\times$ (the group of units of M) to M. This map is always injective, but only surjective if M is a group. The adjunction is therefore not a reflection.

Example 2.1.5, the functors D and U The right adjoint functor U is not full. The given adjunction is therefore not a reflection.

Example 2.1.5, the functors U and I The right adjoint functor I is full and faithful. The given the adjunction is therefore a reflection.

Example 2.1.6 The right adjoint functor $(-)^B$ is not faithful if B is empty, and it is not full if the set B contains at least two distinct elements. It is full and faithful if and only if the set B is a singleton, in which case both the left adjoint functor $(-) \times B$ and the right adjoint functor $(-)^B$ are essentially inverse equivalences of categories.

Exercise 2.2.13

(a)

The induced map $f_* : \mathscr{P}(K) \to \mathscr{P}(L)$ that assigns to each subset of K its image under f is order-preserving, and so can be seen as a functor. The functor f_* is left adjoint to the functor f^* because

$$f(S) \subseteq T \iff f(s) \in T \text{ for every element } s \text{ of } S \iff S \subseteq f^{-1}(T)$$

for every subset S of K and every subset T of L.

To figure out the right adjoint $f_\#$ of f^* we observe that it needs to satisfy

$$t \in f_\#(U) \iff \{t\} \subseteq f_\#(U) \iff f^{-1}(t) \subseteq U$$

for every subset U of K and every element t of K. We therefore set

$$f_\#(U) := \{t \in L \mid f^{-1}(t) \subseteq U\}$$

for every subset U of L. This defines a map $f_\#$ from $\mathscr{P}(K)$ to $\mathscr{P}(L)$ that is order-preserving, and can therefore be regarded as a functor. We have

$$\begin{aligned} T \subseteq f_\#(U) &\iff t \in f_\#(U) \text{ for every element } t \text{ of } T \\ &\iff f^{-1}(t) \subseteq U \text{ for every element } t \text{ of } T \\ &\iff f^{-1}(T) \subseteq U \end{aligned}$$

for every subset T of K and every subset U of L. Therefore, $f_\#$ is right adjoint to f^*.

(b)

The projection map p from $X \times Y$ to X induces a functor

$$p^* : \mathscr{P}(X) \longrightarrow \mathscr{P}(X \times Y).$$

Logically, the action of p^* can be expressed as

$$\begin{aligned} p^*(S)(x, y) &\iff (x, y) \in p^*(S) \\ &\iff (x, y) \in p^{-1}(S) \\ &\iff p(x, y) \in S \\ &\iff x \in S \\ &\iff S(x). \end{aligned}$$

We have seen explicit constructions of the left adjoint functor p_* of p^* and the right adjoint functor $p_\#$ of p^* in the previous part of this exercise. The action of the left adjoint functor p_* can be expressed as

$$\begin{aligned} p_*(R)(x) &\iff x \in p(R) \\ &\iff \text{there exists an element } y \text{ of } Y \text{ with } (x,y) \in R \\ &\iff \text{there exists an element } y \text{ of } Y \text{ with } R(x,y) \\ &\iff \exists y \in Y \,:\, R(x,y)\,. \end{aligned}$$

The action of the right adjoint functor $p_\#$ can be expressed as

$$\begin{aligned} p_\#(R)(x) &\iff x \in p_\#(R) \\ &\iff p^{-1}(x) \subseteq R \\ &\iff (x,y) \in R \text{ for every element } y \text{ of } Y \\ &\iff R(x,y) \text{ for every element } y \text{ of } Y \\ &\iff \forall y \in Y \,:\, R(x,y)\,. \end{aligned}$$

For two subsets S and T of a finite product $X_1 \times \cdots \times X_n$ we have the chain of equivalences

$$\begin{aligned} & S \subseteq T \\ \iff & [(x_1,\dots,x_n) \in S \implies (x_1,\dots,x_n) \in T \text{ for all } (x_1,\dots,x_n) \in X_1 \times \cdots \times X_n] \\ \iff & [S(x_1,\dots,x_n) \implies T(x_1,\dots,x_n) \text{ for all } (x_1,\dots,x_n) \in X_1 \times \cdots \times X_n] \\ \iff & [S \implies T]\,. \end{aligned}$$

The unit η and counit ε of the adjunction $p_* \dashv p^*$ are natural transformations

$$\eta : 1_{\mathscr{P}(X\times Y)} \Longrightarrow p^* p_*\,, \quad \varepsilon : p_* p^* \Longrightarrow 1_{\mathscr{P}(X)}\,,$$

and can therefore be regarded as certain logical implications. The composite $p^* p_*$ can be computed as

$$p^*(p_*(R))(x,y) \iff p_*(R)(x) \iff [\exists y' \,:\, R(x,y')]\,,$$

whence the unit η of the adjunction $f_* \dashv f^*$ can be interpreted as the implication

$$R \implies (\exists y \,:\, R(-,y))\,.$$

The composite $p_* p^*$ can be computed as

$$p_*(p^*(S))(x) \iff [\exists y : p^*(S)(x,y)] \iff [\exists y : S(x)]\,,$$

whence the counit ε of the adjunction $f^* \dashv f_\#$ can be interpreted as the implication

$$[\exists y : S] \implies S\,.$$[11]

We can similarly regard the unit η' and counit ε' of the adjunction $p^* \dashv p_\#$ as logical implications. The composite $p_\# p^*$ can be computed as

$$p_\#(p^*(S))(x) \iff [\forall y \in Y : p^*(S)(x,y)] \iff [\forall y \in Y : S(x)]\,,$$

whence the unit η' can be interpreted as the implication

$$S \implies [\forall y \in Y : S]\,.$$

The composite $p^* p_\#$ can be computed as

$$p^*(p_\#(R))(x,y) \iff p_\#(R)(x) \iff [\forall y' \in Y : R(x,y')]\,,$$

whence the counit ε' can be interpreted as the implication

$$[\forall y \in Y : R(-,y)] \implies R\,.$$

Exercise 2.2.14

Let $\mathscr{S}$ be a category.

1. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories and let F be a functor from $\mathscr{A}$ to $\mathscr{B}$. We make the following observations:
   - Let K be a functor from $\mathscr{B}$ to $\mathscr{S}$. The composite KF is a functor from $\mathscr{A}$ to $\mathscr{S}$.
   - Let K and L be two functors from $\mathscr{B}$ to $\mathscr{S}$ and let α be a natural transformation from K to L. We then have the induced natural transformation αF from KF to LF.
   - The above assignments define a functor F^* from $[\mathscr{B}, \mathscr{S}]$ to $[\mathscr{A}, \mathscr{S}]$. Let us check the functoriality of F^*:

[11] This implication makes sense because the statement S does not depend on the variable y.

- For every functor K from $\mathscr{B}$ to $\mathscr{A}$ we have the equalities

  $$F^*(1_K)_A = (1_K F)_A = (1_K)_{F(A)} = 1_{KF(A)} = (1_{KF})_A = (1_{F^*(K)})_A\,,$$

  for every object A of $\mathscr{A}$, and therefore the equality of natural transformations

  $$F^*(1_K) = 1_{F^*(K)}\,.$$

- Let K, L and M be functors from $\mathscr{B}$ to $\mathscr{S}$ and let

  $$\alpha :\ K \Longrightarrow L\,, \quad \beta :\ L \Longrightarrow M$$

  be two natural transformations. We have the chain of equalities

  $$\begin{aligned}(F^*(\beta) \circ F^*(\alpha))_A &= F^*(\beta)_A \circ F^*(\alpha)_A \\ &= (\beta F)_A \circ (\alpha F)_A \\ &= \beta_{F(A)} \circ \alpha_{F(A)} \\ &= (\beta \circ \alpha)_{F(A)} \\ &= ((\beta \circ \alpha)F)_A \\ &= F^*(\beta \circ \alpha)_A\end{aligned}$$

  for every object A of $\mathscr{A}$, and therefore the equality of natural transformations

  $$F^*(\beta) \circ F^*(\alpha) = F^*(\beta \circ \alpha)\,.$$

We have thus constructed an induced functor F^* from $[\mathscr{B}, \mathscr{S}]$ to $[\mathscr{A}, \mathscr{S}]$.

2. Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be three categories, and let

   $$F :\ \mathscr{A} \longrightarrow \mathscr{B}\,, \quad G :\ \mathscr{B} \longrightarrow \mathscr{C}$$

   be composable functors. We then have the equality of functors

   $$(G \circ F)^* = F^* \circ G^*\,.$$

   Let us check this claimed equality in more detail:

   - Let K be a functor from $\mathscr{C}$ to $\mathscr{S}$. Then

     $$(G \circ F)^*(K) = K \circ G \circ F = G^*(K) \circ F = F^*(G^*(K)) = (F^* \circ G^*)(K)\,.$$

- Let K and L be two functors from $\mathscr{C}$ to $\mathscr{S}$ and let

  $$\alpha : K \Longrightarrow L$$

  be a natural transformation. We have the chain of equalities

  $$\begin{aligned}(G \circ F)^*(\alpha)_A &= (\alpha(G \circ F))_A \\ &= \alpha_{(G \circ F)(A)} \\ &= \alpha_{G(F(A))} \\ &= (\alpha G)_{F(A)} \\ &= ((\alpha G)F)_A \\ &= (G^*(\alpha)F)_A \\ &= F^*(G^*(\alpha))_A \\ &= (F^* \circ G^*)(\alpha)_A\end{aligned}$$

  for every object A of $\mathscr{A}$, and therefore the equality of natural transformations

  $$(G \circ F)^*(\alpha) = (F^* \circ G^*)(\alpha)\,.$$

3. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories and let F and G be two functors from $\mathscr{A}$ to $\mathscr{B}$, and let α be a natural transformation from F to G. For every object K of the functor category $[\mathscr{B}, \mathscr{S}]$, i.e., functor from $\mathscr{B}$ to $\mathscr{S}$, we get an induced natural transformation $K\alpha$ from KF to KG. This induced natural transformation is a morphism from $F^*(K)$ to $G^*(K)$ in the functor category $[\mathscr{A}, \mathscr{S}]$. By setting

   $$(\alpha^*)_K := K\alpha$$

   for every object K of $[\mathscr{B}, \mathscr{S}]$, we therefore arrive at an induced transformation α^* from F^* to G^*.

   The transformation α^* is again natural. To check this, let

   $$\beta : K \longrightarrow L$$

   be a morphism in $[\mathscr{B}, \mathscr{S}]$. This means that K and L are functors from $\mathscr{B}$ to $\mathscr{S}$ and that β is a natural transformation from K to L. We need to check that the diagram

   $$\begin{tikzcd} F^*(K) \arrow[r, "F^*(\beta)"] \arrow[d, "(\alpha^*)_K"'] & F^*(L) \arrow[d, "(\alpha^*)_L"] \\ G^*(K) \arrow[r, "G^*(\beta)"] & G^*(L) \end{tikzcd}$$

commutes. This diagram can be simplified as follows:

$$\begin{tikzcd} KF \arrow[r, "\beta F"] \arrow[d, "K\alpha"'] & LF \arrow[d, "L\alpha"] \\ KG \arrow[r, "\beta G"] & LG \end{tikzcd}$$

This diagram indeed commutes because, by the interchange law for horizontal and vertical composition, both $L\alpha \circ \beta F$ and $\beta G \circ K\alpha$ are given by the horizontal composition $\beta * \alpha$. Indeed, we have the equalities

$$L\alpha \circ \beta F = (1_L * \alpha) \circ (\beta * 1_F) = (1_L \circ \beta) * (\alpha \circ 1_F) = \beta * \alpha$$

and

$$\beta G \circ K\alpha = (\beta * 1_G) \circ (1_K * \alpha) = (\beta \circ 1_K) * (1_G \circ \alpha) = \beta * \alpha\,.$$

We have thus extended the construction $(-)^*$ to natural transformations.

4. The induced natural transformation α^* depends covariantly on α.[12] Let us check this claim:
   - Let F be a functor from $\mathscr{A}$ to $\mathscr{B}$. We have for every object K of $[\mathscr{B}, \mathscr{S}]$ the chain of equalities
   $$((1_F)^*)_K = K1_F = 1_{KF} = 1_{F^*(K)} = (1_{F^*})_K\,,$$
   and therefore the equality of natural transformations $(1_F)^* = 1_{F^*}$.
   - Let now F, G and H be three functors from $\mathscr{A}$ to $\mathscr{B}$, and let
   $$\alpha \colon F \Longrightarrow G\,, \quad \beta \colon G \Longrightarrow H$$
   be two composable natural transformations. Then
   $$((\beta \circ \alpha)^*)_K = K(\beta \circ \alpha) = (K\beta) \circ (K\alpha) = (\beta_*)_K \circ (\alpha^*)_K = (\beta_* \circ \alpha_*)_K$$
   for every object K of $[\mathscr{B}, \mathscr{A}]$, and therefore
   $$(\beta \circ \alpha)^* = \beta^* \circ \alpha^*\,.$$

[12] Our notation of α^* is pretty bad in that regard, since it seems to suggest that α^* depends contravariantly on α.

5. Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be three categories. Let

$$F, G : \mathscr{A} \longrightarrow \mathscr{B}, \quad H : \mathscr{B} \longrightarrow \mathscr{C}$$

be functors, and let

$$\alpha : F \Longrightarrow G$$

be a natural transformation.

We have two ways of obtaining from α and H an induced natural transformation from the functor $(HF)^* = F^*H^*$ to the functor $(HG)^* = G^*H^*$. On the one hand, we can form the natural transformation $H\alpha$ from HF to HG, which then induces the natural transformation $(H\alpha)^*$ from $(HF)^*$ to $(HG)^*$. On the other hand, we can form the induced natural transformation α^* from F^* to G^*, and then consider the resulting natural transformation α^*H^* from F^*H^* to G^*H^*.

These natural transformations turn out to be the same, i.e., we have the equality of natural transformations

$$(H\alpha)^* = \alpha^*H^* .$$

Indeed, we have for every object K of $[\mathscr{C}, \mathscr{S}]$ the chain of equalities

$$((H\alpha)^*)_K = K(H\alpha) = (KH)\alpha = (\alpha^*)_{KH} = (\alpha^*)_{H^*(K)} = (\alpha^*H^*)_K ,$$

and therefore overall the equality $(H\alpha)^* = \alpha^*H^*$.

We are now well-prepared to prove the statement at hand. We consider an adjunction between two categories $\mathscr{A}$ and $\mathscr{B}$ given by two functors

$$F : \mathscr{A} \longrightarrow \mathscr{B}, \quad G : \mathscr{B} \longrightarrow \mathscr{A}$$

and two natural transformations

$$\eta : 1_{\mathscr{A}} \Longrightarrow GF, \quad \varepsilon : FG \Longrightarrow 1_{\mathscr{B}}$$

that serve as the unit and counit of the adjunction respectively. (The functor F is left adjoint to the functor G.) We know that these data satisfy the triangle identities, i.e., that the following two diagrams commute:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd} \tag{2.7}$$

We have seen in above discussions that the functors F and G induce functors

$$F^* : [\mathscr{B}, \mathscr{S}] \longrightarrow [\mathscr{A}, \mathscr{S}], \quad G^* : [\mathscr{A}, \mathscr{S}] \longrightarrow [\mathscr{B}, \mathscr{S}].$$

We have also seen that the natural transformations η and ε induce natural transformations

$$\eta^* : (1_{\mathscr{A}})^* \Longrightarrow (GF)^*, \quad \varepsilon^* : (FG)^* \Longrightarrow (1_{\mathscr{B}})^*.$$

By rewriting the domain and codomain of both η^* and ε^*, we see that these natural transformations are of the forms

$$\eta^* : 1_{[\mathscr{A}, \mathscr{S}]} \Longrightarrow F^* G^*, \quad \varepsilon^* : G^* F^* \Longrightarrow 1_{[\mathscr{B}, \mathscr{S}]}.$$

We can dualize the diagrams (2.7) to get the commutative diagrams

$$\begin{tikzcd} F^* \arrow[r, "(F\eta)^*"] \arrow[dr, "(1_F)^*"'] & (FGF)^* \arrow[d, "(\varepsilon F)^*"] \\ & F^* \end{tikzcd} \qquad \begin{tikzcd} G^* \arrow[r, "(\eta G)^*"] \arrow[dr, "(1_G)^*"'] & (GFG)^* \arrow[d, "(G\varepsilon)^*"] \\ & G* \end{tikzcd} \tag{2.8}$$

These diagrams can be simplified as follows:

$$\begin{tikzcd} F^* \arrow[r, "\eta^* F^*"] \arrow[dr, "1_{F^*}"'] & F^* G^* F^* \arrow[d, "F^* \varepsilon^*"] \\ & F^* \end{tikzcd} \qquad \begin{tikzcd} G^* \arrow[r, "G^* \eta^*"] \arrow[dr, "1_{G^*}"'] & G^* F^* G^* \arrow[d, "\varepsilon^* G^*"] \\ & G* \end{tikzcd} \tag{2.9}$$

The commutativity of these diagrams tells us that the natural transformations η^* and ε^* serve as the unit and counit of an adjunction between the categories $[\mathscr{A}, \mathscr{S}]$ and $[\mathscr{B}, \mathscr{S}]$, with F^* right adjoint to G^*.

2.3 Adjunctions via initial objects

Exercise 2.3.8

Let G and H be two groups, and let $\mathscr{G}$ and $\mathscr{H}$ be the corresponding one-object categories. We make the following observations:

- A functor from $\mathscr{G}$ to $\mathscr{H}$ is the same as a homomorphism of groups from G to H.
- Given two functors F and F' from $\mathscr{G}$ to $\mathscr{H}$, every natural transformation from F to F' is already a natural isomorphism since every morphism in $\mathscr{H}$ is an isomorphism.

  Such a natural isomorphism from F to F' consists of a single component, which is an element h of H with $hFh^{-1} = F'$. This means in particular that there exists a natural transformation between the functors F and F' if and only if F and F' are conjugated as homomorphisms of groups.

Suppose now that $(L, R, \eta, \varepsilon)$ is an adjunction between the categories $\mathscr{G}$ and $\mathscr{H}$. This adjunction consists of two functors

$$L \colon \mathscr{G} \longrightarrow \mathscr{H}\,, \quad R \colon \mathscr{H} \longrightarrow \mathscr{G}\,,$$

and two natural transformations

$$\eta \colon 1_{\mathscr{G}} \Longrightarrow RL\,, \quad \varepsilon \colon LR \Longrightarrow 1_{\mathscr{H}}$$

that serve as the unit and the counit of the adjunction respectively. The functors L and R correspond to homomorphisms of groups

$$l \colon G \longrightarrow H\,, \quad r \colon H \longrightarrow G\,.$$

Both η and ε are actually natural isomorphisms because all morphisms in $\mathscr{G}$ and all morphisms in $\mathscr{H}$ are isomorphisms. It follows from Exercise 2.2.11 that both L and R are equivalences of categories. This entails that these functors are fully faithful, which means that l and r are bijective, and therefore isomorphisms of groups.

The natural transformation η consists of only a single component, which is an element g of G. That η is a natural transformation from $1_{\mathscr{G}}$ to the composite RL means that the diagram

$$\begin{tikzcd}
*_{\mathscr{G}} \arrow[r, "g'"] \arrow[d, "g"'] & *_{\mathscr{G}} \arrow[d, "g"] \\
*_{\mathscr{G}} \arrow[r, "r(l(g'))"] & *_{\mathscr{G}}
\end{tikzcd}$$

commutes for every element g' of G. In other words, we have

$$(r \circ l)(g) = gg'g^{-1}$$

for every element g' of G, so that the composite $r \circ l$ is the inner automorphism of G given by conjugation with g.

We find similarly that the natural transformation ε consists of only a single component, which an element h of H, and that the composite $l \circ r$ is the inner automorphism of H given by conjugation with h^{-1}.

That the natural transformations η and ε define an adjunction between the functors L and R is equivalent to the triangle identities, i.e., to the commutativity of the following two diagrams:

$$\begin{tikzcd} L \arrow[r, "L\eta"] \arrow[dr, "1_L"'] & LRL \arrow[d, "\varepsilon L"] \\ & L \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\eta R"] \arrow[dr, "1_R"'] & RLR \arrow[d, "R\varepsilon"] \\ & R \end{tikzcd}$$

By evaluating these diagrams at the single objects of the categories $\mathscr{G}$ and $\mathscr{H}$, they can equivalently be expressed as follows:

$$\begin{tikzcd} *_{\mathscr{H}} \arrow[r, "l(g)"] \arrow[dr, "1_H"'] & *_{\mathscr{H}} \arrow[d, "h"] \\ & *_{\mathscr{H}} \end{tikzcd} \qquad \begin{tikzcd} *_{\mathscr{G}} \arrow[r, "g"] \arrow[dr, "1_G"'] & *_{\mathscr{G}} \arrow[d, "r(h)"] \\ & *_{\mathscr{G}} \end{tikzcd}$$

The commutativity of these diagrams is equivalent two the two conditions

$$h = l(g)^{-1}, \quad g = r(h)^{-1}.$$

We find overall that an adjunction between two groups G and H, when regarded as one-object categories, consists of two isomorphisms of groups

$$l \colon G \longrightarrow H, \quad r \colon H \longrightarrow G,$$

and elements g and h of G and H respectively, subject to the following conditions: the composite $r \circ l$ is given by conjugation with g, the composite $r \circ l$ is given by composition with h^{-1}, and the two elements g and h are related via $h = l(g)^{-1}$ and $g = r(h)^{-1}$.

Exercise 2.3.9

The dual statement reads as follows:

> Let $\mathscr{A}$ and $\mathscr{B}$ be two categories, and let F be a functor from $\mathscr{A}$ to $\mathscr{B}$. Then F has a right adjoint if and only if for every object B of $\mathscr{B}$, the category $F \Rightarrow B$ has a terminal object.

To prove, the statement, we regard F as a functor F' from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{B}^{\mathrm{op}}$. A functor from $\mathscr{B}$ to $\mathscr{A}$ is right adjoint to F if and only if, as a functor from $\mathscr{B}^{\mathrm{op}}$ to $\mathscr{A}^{\mathrm{op}}$, it is left adjoint to F'. (See Proposition 2.A (page 62).) It follows that the functor F admits a right adjoint if and only if the functor F' admits a left adjoint.

According to Corollary 2.3.7, this is the case if and only if for every object B of $\mathscr{B}$, the category $B^{\mathrm{op}} \Rightarrow F'$ admits an initial object. The category $B^{\mathrm{op}} \Rightarrow F'$ is isomorphic to the category $(F \Rightarrow B)^{\mathrm{op}}$. Therefore, the category $B^{\mathrm{op}} \Rightarrow F'$ admits an initial object if and only if the category $F \Rightarrow B$ admits a terminal object.

This shows the claim that the functor F admits a right adjoint if and only if for every object B of $\mathscr{B}$, the category $F \Rightarrow B$ admits a terminal object.

Exercise 2.3.10

For every object A of $\mathscr{A}$ and every object B of $\mathscr{B}$, let $\Phi_{A,B}$ be the composite

$$\mathscr{A}(A, G(B)) \xrightarrow{F} \mathscr{B}(F(A), FG(B)) \xrightarrow{(\varepsilon_B)_*} \mathscr{B}(F(A), B)\,.$$

The functor F is full and faithful and the morphism ε_B is an isomorphism. The map $\Phi_{A,B}$ is therefore a composite of two bijections, and thus again a bijection. We show in the following that the bijection $\Phi_{A,B}$ is natural in both A and B.

For the naturality of A, we note that for every morphism

$$f \colon A \longrightarrow A'$$

in $\mathscr{A}$, we have the following diagram:

$$\begin{tikzcd}
\mathscr{A}(A, G(B)) \arrow[r, "F"] & \mathscr{B}(F(A), FG(B)) \arrow[r, "(\varepsilon_B)_*"] & \mathscr{B}(F(A), B) \\
\mathscr{A}(A', G(B)) \arrow[r, "F"] \arrow[u, "f^*"] & \mathscr{B}(F(A'), FG(B)) \arrow[r, "(\varepsilon_B)_*"] \arrow[u, "F(f)^*"'] & \mathscr{B}(F(A'), B) \arrow[u, "F(f)^*"']
\end{tikzcd}$$

The left part of this diagram commutes by the functoriality of F. The right side also commutes. It follows that the entire diagram commutes, which entails that the following, outer diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(A, G(B)) \arrow[r, "\Phi_{A,B}"] & \mathscr{B}(F(A), B) \\
\mathscr{A}(A', G(B)) \arrow[u, "f^*"] \arrow[r, "\Phi_{A',B}"] & \mathscr{B}(F(A'), B) \arrow[u, "F(f)^*"']
\end{tikzcd}$$

This proves the desired naturality.

For the naturality in B, we note that for every morphism

$$g : B \longrightarrow B'$$

in $\mathscr{B}$, we have the following diagram:

$$\begin{tikzcd}
\mathscr{A}(A, G(B)) \arrow[r, "F"] \arrow[d, "G(g)_*"'] & \mathscr{B}(F(A), FG(B)) \arrow[r, "(\varepsilon_B)_*"] \arrow[d, "FG(g)_*"] & \mathscr{B}(F(A), B) \arrow[d, "g_*"] \\
\mathscr{A}(A, G(B')) \arrow[r, "F"] & \mathscr{B}(F(A), FG(B')) \arrow[r, "(\varepsilon_{B'})_*"] & \mathscr{B}(F(A), B')
\end{tikzcd}$$

The left side of this diagram commutes by the functoriality of F. The right side commutes by the naturality of ε. It follows that the entire diagram commutes, which entails that the following outer diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(A, G(B)) \arrow[r, "\Phi_{A,B}"] \arrow[d, "G(g)_*"'] & \mathscr{B}(F(A), B) \arrow[d, "g_*"] \\
\mathscr{A}(A, G(B')) \arrow[r, "\Phi_{A,B'}"] & \mathscr{B}(F(A), B')
\end{tikzcd}$$

This shows the desired naturality.

We have thus constructed an adjunction between the two functors F and G, with F left-adjoint to G. (We have constructed this adjunction in precisely such a way that ε is its counit; we could have also constructed an adjunction for which η is its unit.)

Exercise 2.3.11

There exists by assumption two elements a_1 and a_2 of $U(A)$ that are distinct.

Let S be some set and let s_1 and s_2 be any two distinct elements of S. We consider the set-theoretic map

$$f : S \longrightarrow U(A)$$

given by $f(s_1) := a_1$, and $f(s) := a_2$ for every element s of S with $s \neq a_1$. This function corresponds to a morphism $\overline{f}$ from $F(S)$ to A, and the function f can be retrieved from the morphism $\overline{f}$ via

$$f = U(\overline{f}) \circ \eta_S \,.$$

It follows from the chain of relations

$$U(\overline{f})(\eta_S(s_1)) = f(s_1) = a_1 \neq a_2 = f(s_2) = U(\overline{f})(\eta_S(s_2))$$

that also $\eta_S(s_1) \neq \eta_S(s_2)$. We have thus shown that the map η_S is injective.

There exists plenty of groups whose underlying set consists of at least two elements. It follows from the above discussion that for every set S, the canonical map from S into the free group of S is injective.

Exercise 2.3.12

Equivalence of categories between Par and $\mathbf{Set}_*$

In the following, we are sometimes presented with the following situation: we are given a partially defined function (S, f), a subset T of S, and we wish to consider the restriction of (S, f) to T. The proper notation for this restriction is $(S, f|_T)$.

$$(T, f|_T) \,.$$

For better readability, we sometimes denote this restriction as

$$(T, f)$$

instead. This shan’t lead to confusion, since the domain T is still part of the notation.

We define a functor F from **Par** to $\mathbf{Set}_*$ as follows.

- For every set X, let $F(X)$ be the pointed set that arises from X by adjoining a base point x_0. (More explicitly: let x_0 be an element not contained in X, and then let $F(X)$ be the pointed set $(X \cup \{x_0\}, x_0)$. From a set-theoretic perspective, this element x_0 may be chosen to be the set X itself, since no set can be an element of itself.)
- Let (S, f) be a partially defined function from a set X to a set Y. Let $F((S, f))$ be the pointed map from $F(X)$ to $F(Y)$ given by

$$F((S,f)) : \; x \longmapsto \begin{cases} f(x) & \text{if } x \in S, \\ y_0 & \text{otherwise.} \end{cases}$$ [13]

These assignments define a functor from **Par** to $\mathbf{Set}_*$:

- Let X be a set. The identity morphism of X in the category **Par** is the pair $(X, 1_X)$, where 1_X denotes the identity map on the set X. The induced map $F((X, 1_X))$ sends every element of X to itself, and also the basepoint x_0 to itself. In other words, the map $F((X, 1_X))$ is the identity map on the set $F(X)$. In formula, $F((X, 1_X)) = 1_{F(X)}$.
- Let X, Y and Z be three sets, and let

$$(S, f) : \; X \longrightarrow Y \,, \quad (T, g) : \; Y \longrightarrow Z$$

  be two partially defined functions. The composite $F((T, g)) \circ F((S, f))$ is given on elements of X as follows:

  - The basepoint x_0 is mapped to the basepoint y_0 by $F((S, f))$, and then further to the basepoint z_0 by $F((T, g))$.
  - Let x be an element of X not contained in S. The element x is first mapped to the basepoint y_0 by $F((S, f))$, and then further mapped to the basepoint z_0 by $F((T, g))$.
  - Let x be an element of X that is contained in the domain S of f, but not in the preimage $f^{-1}(T)$. This element x is first mapped to the point $f(x)$ in Y by $F((S, f))$. The element $f(x)$ lies outside T, whence it is then further mapped to the base point z_0 by $F((T, g))$.

[13] One might think about the newly added base point y_0 of Y as the value “`undefined`”. The map $F((S, f))$ is then an extension of the partially defined function (S, f) to a map

$$X \cup \{\texttt{undefined}\} \longrightarrow Y \cup \{\texttt{undefined}\}$$

that maps all input values outside S to `undefined`.

- ◦ Let x be an element of X that is not only contained in S, but also in the preimage $f^{-1}(T)$. This element x is first mapped to the point $f(x)$ in T by $F((S,f))$, and then further mapped to the point $g(f(x))$ by $F((T,g))$.

  We find overall that the composite $F((T,g)) \circ F((S,f))$ coincides with the induced map $F((T,g) \circ (S,f))$.[14]

The functor G from $\mathbf{Set}_*$ to $\mathbf{Par}$, which will serve as an essential inverse to F, is easier to construct:

- Given a pointed set (X, x_0), its image under G is given by the set $X \setminus \{x_0\}$.
- Given a pointed map
  $$f : (X, x_0) \longrightarrow (Y, y_0),$$
  its image under G is given by the restriction of f to the set $X \setminus g^{-1}(y_0)$, i.e., the partially defined function $(X \setminus f^{-1}(y_0), f)$. The domain $X \setminus f^{-1}(y_0)$ is a subset of $X \setminus \{x_0\}$ because the basepoint x_0 is contained in the preimage $f^{-1}(y_0)$.

This assignment G is indeed a functor from $\mathbf{Set}_*$ to $\mathbf{Par}$:

- Let (X, x_0) be a pointed set. Its identity morphism in the category $\mathbf{Set}_*$ is simply the identity map of the set X, i.e.,
  $$1_{(X,x_0)} = 1_X .$$
  The resulting map $G(1_{(X,x_0)})$ is the restriction of 1_X to the domain
  $$X \setminus 1_X^{-1}(x_0) = X \setminus \{x_0\}.$$
  Therefore,
  $$G(1_{(X,x_0)}) = (X \setminus \{x_0\}, 1_X) = (X \setminus \{x_0\}, 1_{X \setminus \{x_0\}}) = 1_{G((X,x_0))} .$$
- Let
  $$f : (X, x_0) \longrightarrow (Y, y_0), \quad g : (Y, y_0) \longrightarrow (Z, z_0)$$
  be two composable maps of pointed sets. We have the equality of sets
  $$(g \circ f)^{-1}(z_0) = f^{-1}(g^{-1}(z_0)),$$

[14] The composite $(T,g) \circ (S,f)$ is the partially defined function $(f^{-1}(T), g \circ f)$.

and therefore the equality of partially defined functions

$$\begin{aligned} G(g) \circ G(f) &= (Y \setminus g^{-1}(z_0), g) \circ (X \setminus f^{-1}(y_0), f) \\ &= (f^{-1}(Y \setminus g^{-1}(z_0)), g \circ f) \\ &= (f^{-1}(Y) \setminus f^{-1}(g^{-1}(z_0)), g \circ f) \\ &= (X \setminus (g \circ f)^{-1}(z_0), g \circ f) \\ &= G(g \circ f) . \end{aligned}$$

In the following, we will show that the two functors F and G form an equivalence of categories between **Par** and **Set**$_*$. We do so by constructing natural isomorphisms

$$\alpha : 1_{\mathbf{Par}} \Longrightarrow GF , \quad \beta : 1_{\mathbf{Set}_*} \Longrightarrow FG .$$

We observe first that $GF = 1_{\mathbf{Par}}$. This allows us to choose α as the identity natural transformation.

To construct the natural isomorphism β, let (X, x_0) be a pointed set. The pointed set $FG((X, x_0))$ is given by

$$(X \setminus \{x_0\} \cup \{x_0'\}, \ x_0')$$

with $x_0' \notin X \setminus \{x_0\}$. In other words: we first remove the base point x_0 of X, and then add a new basepoint x_0' in its place. (We can, however, not ensure that the newly added base point x_0' is set-theoretically equal to the previous base point x_0. This is why the composite FG won't be the identity functor of **Set**$_*$, but will only be isomorphic to it.) The map

$$\beta_{(X,x_0)} : (X, x_0) \longrightarrow (X \setminus \{x_0\} \cup \{x_0'\}, \ x_0')$$

given by

$$x \longmapsto \begin{cases} x_0' & \text{if } x = x_0, \\ x & \text{otherwise,} \end{cases}$$

is therefore an isomorphism of pointed sets from (X, x_0) to $FG((X, x_0))$.

Let us show the naturality of β. For this, let

$$f : (X, x_0) \longrightarrow (Y, y_0) ,$$

be a pointed map. The domain and codomain $GF(f)$ are given by

$$GF((X, x_0)) = X_0 \setminus \{x_0\} \cup \{x_0'\} , \quad GF((Y, y_0)) = Y_0 \setminus \{y_0\} \cup \{y_0'\}$$

respectively. The pointed map $GF(f)$ is given by

$$GF(f)(x) = \begin{cases} y_0' & \text{if } x = x_0', \\ f(x) & \text{otherwise,} \end{cases}$$

for every element x of $GF((X, x_0)) = X_0 \setminus \{x_0\} \cup \{x_0'\}$. The diagram

$$\begin{tikzcd} (X, x_0) \arrow[r, "f"] \arrow[d, "\beta_{(X,x_0)}"'] & (Y, y_0) \arrow[d, "\beta_{(Y,y_0)}"] \\ GF((X, x_0)) \arrow[r, "GF(f)"] & GF((Y, y_0)) \end{tikzcd}$$

therefore commutes, showing the naturality of β.

We have overall constructed a natural isomorphism β from $1_{\mathbf{Set}_*}$ to GF.

Description of Set_* as a coslice category

We can describe the category $\mathbf{Set}_*$ as the coslice category $\mathbb{1}/\mathbf{Set}$, where $\mathbb{1} := \{*\}$ is an one-element set.

- An object of the category $\mathbb{1}/\mathbf{Set}$ is a pair (X, ξ) consisting of a set X and a map ξ from $\mathbb{1}$ to X. This map ξ amounts to picking out an element x_0 of X, namely the element $\xi(*)$.
- A morphism
$$f : (X, \xi) \longrightarrow (Y, \upsilon)$$
in $\mathbf{Set}_*$ is a set-theoretic map f from X to Y that makes the diagram
$$\begin{tikzcd} & \mathbb{1} \arrow[dl, "\xi"'] \arrow[dr, "\upsilon"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$
commute. The commutativity of this diagram can be expressed in terms of the points $x_0 := \xi(*)$ and $y_0 := \upsilon(*)$ as the equality
$$f(x_0) = y_0 .$$
That means that a morphism from (X, ξ) to (Y, υ) is the same as a pointed map from (X, x_0) to (Y, y_0).

- The composition of morphisms is $\mathbf{Set}_*$ is the usual composition of functions, and the same goes for the coslice category $\mathbb{1}/\mathbf{Set}$.

We can see from these observations that the two categories $\mathbf{Set}_*$ and $\mathbb{1}/\mathbf{Set}$ are isomorphic.

Chapter 3

Interlude on sets

3.1 Constructions with sets

Exercise 3.1.1

Left adjoint

For any two sets A and B, let i_A and j_B denote the inclusion maps from A and B to $A + B$ respectively.

There exists for every pair (f, g) of maps

$$f : A \longrightarrow A', \quad g : B \longrightarrow B'$$

a unique induced map

$$f + g : A + B \longrightarrow A' + B',$$

such that

$$(f + g) \circ i_A = f \quad \text{and} \quad (f + g) \circ j_B = g \,.$$[1]

We have for every two sets A and B the equality

$$1_A + 1_B = 1_{A+B},$$

[1] If one thinks about the sum $A + B$ as the disjoint union of the sets A and B, then the function $f + g$ is given for every element x of $A + B$ by

$$(f + g)(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B. \end{cases}$$

and for all maps of the form

$$f : A \longrightarrow A' , \quad f' : A' \longrightarrow A'' , \quad g : B \longrightarrow B' , \quad g' : B' \longrightarrow B'' ,$$

the equality

$$(f' + g') \circ (f + g) = (f' \circ f) + (g' \circ g) .$$

This means that we have a functor

$$S : \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set}$$

that is given by

$$S(A, B) \coloneqq A + B \quad \text{and} \quad S(f, g) = f + g$$

for every object (A, B) of $\mathbf{Set} \times \mathbf{Set}$ and every morphism (f, g) of $\mathbf{Set} \times \mathbf{Set}$.

We have for every three sets A, B and C a bijection

$$\mathbf{Set}(A + B, C) \longrightarrow \mathbf{Set}(A, C) \times \mathbf{Set}(B, C) ,$$

given by

$$h \longmapsto (h \circ i_A, h \circ j_B) .$$

The codomain of this bijection can be rewritten as

$$\begin{aligned} \mathbf{Set}(A, C) \times \mathbf{Set}(B, C) &= (\mathbf{Set} \times \mathbf{Set})((A, B), (C, C)) \\ &= (\mathbf{Set} \times \mathbf{Set})((A, B), \Delta(C)) , \end{aligned}$$

and its domain as

$$\mathbf{Set}(S(A, B), C) .$$

We have thus constructed a bijection

$$\mathbf{Set}(S(A, B), C) \longrightarrow (\mathbf{Set} \times \mathbf{Set})((A, B), \Delta(C)) .$$

It can be checked that this bijection is natural in both (A, B) and C. The functor S is therefore left adjoint to the diagonal functor Δ.

Right adjoint

Every pair (f, g) of maps

$$f : A \longrightarrow A', \quad g : B \longrightarrow B'$$

induces a map

$$f \times g : A \times B \longrightarrow A' \times B'$$

given by

$$(f \times g)(a, b) \coloneqq (f(a), g(b))$$

for every element (a, b) of $A \times B$. We have for every two sets A and B the equality

$$1_A \times 1_B = 1_{A \times B},$$

and for any maps of the form

$$f : A \longrightarrow A', \quad f' : A' \longrightarrow A'', \quad g : B \longrightarrow B', \quad g' : B' \longrightarrow B'',$$

the equality

$$(f' \times g') \circ (f \times g) = (f' \circ f) \times (g' \circ g).$$

This means that we have a functor

$$P : \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set}$$

that is given by

$$P(A, B) \coloneqq A \times B, \quad P(f, g) = f \times g$$

for every object (A, B) of $\mathbf{Set} \times \mathbf{Set}$ and every morphism (f, g) of $\mathbf{Set} \times \mathbf{Set}$.

We have for every three sets A, B and C a bijection

$$\mathbf{Set}(A, B) \times \mathbf{Set}(A, C), \longrightarrow \mathbf{Set}(A, B \times C)$$

given by

$$(f, g) \longmapsto [a \longmapsto (f(a), g(a))]$$

for every element (f, g) of $\mathbf{Set}(A, B) \times \mathbf{Set}(A, C)$. The domain of this bijection can be rewritten as

$$\begin{aligned} \mathbf{Set}(A, B) \times \mathbf{Set}(A, C) &= (\mathbf{Set} \times \mathbf{Set})((A, A), (B, C)) \\ &= (\mathbf{Set} \times \mathbf{Set})(\Delta(A), (B, C)), \end{aligned}$$

and its codomain as

$$\mathbf{Set}(A, P(B, C)) .$$

We have thus constructed a bijection

$$(\mathbf{Set} \times \mathbf{Set})(\Delta(A), (B, C)) \longrightarrow \mathbf{Set}(A, P(B, C)) .$$

It can be checked that this bijection is natural in both A and (B, C). The functor P is therefore right adjoint to the diagonal functor Δ.

Exercise 3.1.2

An object of $\mathscr{C}$ is a triple (X, x_0, f) consisting of a set X, an element x_0 of X, and a function f from X to X. Given two such objects (X, x_0, f) and (Y, y_0, g), a morphism from (X, x_0, f) to (Y, y_0, g) in $\mathscr{C}$ is a set-theoretic map φ from X to Y that satisfies the condition

$$\varphi(x_0) = y_0$$

and that makes the square diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

commute. The composition of two such morphisms is the usual composition of functions.

3.2 Small and large categories

Exercise 3.2.12

(a)

We replace the power set $\mathscr{P}(A)$ by an arbitrary complete lattice P, i.e., by a partially ordered set in which every subset admits a supremum.

Let R be the set of all elements of P that are increased by θ, i.e.,

$$R := \{r \in P \mid r \leq \theta(r)\} .$$

We observe that the set R is closed under the action of θ. Indeed, let r be any element of R. It follows from the inequality $r \leq \theta(r)$ that $\theta(r) \leq \theta(\theta(r))$ because the map θ is order-preserving. This inequality $\theta(r) \leq \theta(\theta(r))$ tells us that the element $\theta(r)$ is again contained in the set R.

Let s be the supremum of R. We show in the following that s is a fixed point of θ. That is, we need to show that $\theta(s) = s$. We will do so by showing that both $s \leq \theta(s)$ and $\theta(s) \leq s$.

We observe that the second inequality follows from the first. Indeed, suppose for a moment that $s \leq \theta(s)$. This inequality tells us that the supremum s is itself again contained in the set R. It then follows that $\theta(s)$ is also contained in R, because R is closed under θ. But s is the supremum of R, so that $r \leq s$ for every element r of R. We may choose r as $\theta(s)$, and thus find that $\theta(s) \leq s$.

We are left to show the inequality $s \leq \theta(s)$. We note that

$$\sup \theta(X) \leq \theta(\sup X)$$

for every subset X of P because the map θ is order-preserving. By choosing for X the set R, we find that

$$s = \sup R = \sup_{r \in R} r \leq \sup_{r \in R} \theta(r) = \sup \theta(R) \leq \theta(\sup R) = \theta(s) \,,$$

where we used once again that the map θ is order-preserving and that the set R is closed under the action of θ.

(b)

We are tasked to show that there exists a subset S of A with

$$g(B \smallsetminus f(S)) = A \smallsetminus S \,.$$

By taking the complements of both side of this equality, we can express the equality equivalently as

$$S = A \smallsetminus g(B \smallsetminus f(S)) \,.$$

We hence need to show that the map

$$\theta \colon \mathscr{P}(A) \longrightarrow \mathscr{P}(A) \,, \quad S \longmapsto A \smallsetminus g(B \smallsetminus f(S))$$

admits a fixed point. According to part (a) of this exercise, it suffices to show that θ is order-preserving. To this end, we note that θ is the composite of the four maps

$$S \longmapsto f(S)\,, \quad T \longmapsto B \setminus T\,, \quad U \longmapsto g(U)\,, \quad V \longmapsto A \setminus V\,.$$

The first and third of these maps are order-preserving, and the second and fourth are order-reversing. It follows that θ is order-preserving, as desired.

(c)

Let A and B be two sets with $|A| \leq |B|$ and $|B| \leq |A|$. By definition, this means that there exist injective functions

$$f \colon A \longrightarrow B \quad \text{and} \quad g \colon B \longrightarrow A\,.$$

There exists by the previous part of this exercise a subset S of A with

$$g(B \setminus f(S)) = A \setminus S\,. \tag{3.1}$$

Let T be the image of S under f, i.e., let $T \coloneqq f(S)$. The injection f restricts to a bijection between the sets S and T, and formula (3.1) tells us that the injection g restricts to a bijection between $B \setminus T$ and $A \setminus S$. By combining these two bijections, we arrive at a bijection between the sets A and B.

The existence of such a bijection shows that A and B have the same cardinality.

Exercise 3.2.13

(a)

We denote the given subset of A by S. Suppose that there exists an element a of A with $S = f(a)$. We may wonder if the element a is contained in the set S. By the definition of S, we have the equivalence

$$a \in S \iff a \notin f(a)\,.$$

By the choice of a, we have the equivalence

$$a \notin f(a) \iff a \notin S\,.$$

By combining both of these equivalences, we arrive at the contradiction

$$a \in S \iff a \notin S\,.$$

We hence find that the desired element a cannot exist. Therefore, f cannot be surjective.

(b)

We know that $|A| \leq |\mathscr{P}(A)|$ because there exists an injective function from A to $\mathscr{P}(A)$, giving by the mapping $a \mapsto \{a\}$.

But we have seen in part (a) of this exercise that there exists no surjective function from A to $\mathscr{P}(A)$. This entails that there exists no bijection between A and $\mathscr{P}(A)$, so that $|A| \neq |\mathscr{P}(A)|$.

By combining both of these observations, we find that $|A| < |\mathscr{P}(A)|$.

Exercise 3.2.14

(a)

Let $(A_i)_{i\in I}$ be a family of objects of $\mathscr{A}$, where I is some index set. We are tasked with finding an object A of $\mathscr{A}$ that is isomorphic to none of the objects A_i.

It suffices to find an object A of $\mathscr{A}$ for which the set $U(A)$ is non-isomorphic to each of sets $U(A_i)$, since the functor U preserves isomorphisms. We can ensure these non-isomorphisms by making the set $U(A)$ have strictly larger cardinality than each of the sets $U(A_i)$. To ensure that $U(A)$ has strictly larger cardinality than $U(A_i)$ for every index i at the same time, we will choose the object A so that $U(A)$ has strictly larger cardinality than the set $\sum_{i\in I} U(A_i)$.[2] To summarize our discussion: we need to find an object A of $\mathscr{A}$ such that

$$\left|\sum_{i\in I} U(A_i)\right| < |U(A)|\,.$$

The functor U is part of an adjunction $(F, U, \eta, \varepsilon)$ between the categories $\mathscr{A}$ and **Set** (by assumption). We know from Exercise 2.3.11 that under the given assumptions, the map

$$\eta_P : P \longrightarrow UF(P)$$

[2] Here we use that I is a set, to ensure that the sum $\sum_{i\in I} A_i$ exists.

is injective for every set P. We have therefore the inequality

$$|P| \leq |UF(P)|$$

for every set P. If we choose the set P to be of strictly larger cardinality than the set $\sum_{i \in I} U(A_i)$ (e.g., its power set), then it follows for the object $A := F(P)$ of $\mathscr{A}$ that

$$\left| \sum_{i \in I} U(A_i) \right| \leq |P| \leq |UF(P)| = |U(A)| .$$

We have thus found the desired object A.

Let us summarize our findings: given a set P of strictly larger cardinality than the sum $\sum_{i \in I} U(A_i)$, we have for the object $A := F(P)$ the chain of inequalities

$$|U(A_j)| \leq \left| \sum_{i \in I} U(A_i) \right| < |P| \leq |UF(P)| = |U(A)|$$

for every index j. These inequalities show that each set $U(A_j)$ is non-isomorphic to the set $U(A)$, whence each object A_j is non-isomorphic to the object A.

(b)

Suppose that the category $\mathscr{A}$ is essentially small. This means that there exists a small category $\mathscr{B}$ equivalent to $\mathscr{A}$. Such an equivalence between $\mathscr{A}$ and $\mathscr{B}$ induces a bijection between the class of isomorphism classes of objects of $\mathscr{A}$ and the class of isomorphism classes of objects of $\mathscr{B}$. (In other words, a bijection between the quotient classes $\mathscr{A}/{\cong}$ and $\mathscr{B}/{\cong}$.) It follows that the category $\mathscr{B}$ again satisfies the assumption of part (a) of this exercise. But the category $\mathscr{B}$ is small, whence the family of objects $(B)_{B \in \mathrm{Ob}(\mathscr{B})}$ is indexed by a set. This family contains all objects of $\mathscr{B}$, contradicting part (a) of this exercise.

(c)

We have for each of the given categories a forgetful functor to **Set**, which then admits a left adjoint:

- The forgetful functor from **Set** to **Set** is the identity functor of **Set**, whose left adjoint is again the identity functor.
- For the algebraic categories $\mathbf{Vect}_{\mathbb{k}}$, **Grp**, **Ab**, and **Ring**, the respective left adjoint functors assigns to any set X the respective free algebraic structures on X.
- For the forgetful functor from **Top** to **Set**, its left adjoint assigns to each set X the discrete topological space whose underlying set is X.

Each of categories **Set**, $\mathbf{Vect}_{\mathbb{k}}$, **Grp**, **Ab**, **Ring** and **Top** contains an object whose underlying set consists of at least two distinct elements. These categories hence satisfy the assumption of part (a) of this exercise. It follows from part (b) of this exercise that these categories are not essentially small.

Exercise 3.2.15

(a)

We can use the strategy developed in Exercise 2.3.14 to see that the category **Mon** is not essentially small. It is therefore also not small. It is, however, locally small, since it admits a faithful functor to the locally small category **Set**.

(b)

The given category is small because its collection of morphisms is the underlying set of $\mathbb{Z}$. This entails that the given category is both essentially small and locally small.

(c)

The given category is locally small because there exists at most one morphism between any two objects of its objects. Its class of objects is the underlying set of $\mathbb{Z}$, and therefore a set. It follows that this category is small, and therefore also essentially small.

(d)

We can regard every set as a discrete category. This allows us to regard **Set** as a full subcategory of **Cat**. Two sets are isomorphic if and only if they are isomorphic as discrete categories. The inclusion from **Set** and **Cat** does therefore induce an injection

$$\mathbf{Set}/{\cong} \longrightarrow \mathbf{Cat}/{\cong}\,.$$

It follows that if **Cat** were to be essentially small, then **Set** would also be essentially small. But we know from Exercise 3.2.14 that this is not the case.

We therefore find that **Cat** is not essentially small, and hence also not small.

The category **Cat** is, however, locally small. To see this, let $\mathscr{A}$ and $\mathscr{B}$ be two objects of **Cat**. This means that both $\mathscr{A}$ and $\mathscr{B}$ are small categories. Every functor from $\mathscr{A}$ to $\mathscr{B}$ is uniquely determined by its action on the morphisms of $\mathscr{A}$, whence the class $\mathbf{Cat}(\mathscr{A},\mathscr{B})$ can be identified with a subset of the set $\mathrm{Mor}(\mathscr{B})^{\mathrm{Mor}(\mathscr{A})}$.[3] This entails that $\mathbf{Cat}(\mathscr{A},\mathscr{B})$ is again a set.

(e)

There exists for every family of cardinals $(\kappa_i)_{i\in I}$ a cardinal λ that is distinct to each κ_i. (One may take for λ the cardinal associated to the power set of the sum $\sum_{i\in I} \kappa_i$. This cardinal is then of strictly larger cardinality than each κ_i.) The collection of all cardinals is therefore a proper class, and not a set. It follows that the given category is not locally small. It is therefore not essentially small, and hence also not small.

Exercise 3.2.16

The functor D

We can regard every set A as a discrete category, which we then denote by $D(A)$. Every set-theoretic function

$$f \colon A \longrightarrow B$$

can then be regarded as a functor $D(f)$ from $D(A)$ to $D(B)$. We arrive in this way at a functor D from **Set** to **Cat**. This functor is left adjoint to the functor O.

[3] We denote here for every category $\mathscr{B}$ the class of morphisms in $\mathscr{C}$ by $\mathrm{Mor}(\mathscr{C})$.

The functor I

We can similarly regard every set A is an indiscrete category, which we will then denote by $I(A)$. The objects of $I(A)$ are the elements of A, and there exists for any two elements a and a' of A precisely one morphism from a to a' in $I(A)$. The composition of morphisms in $I(A)$ is defined in the only possible way.[4] Every set-theoretic function

$$f : A \longrightarrow B$$

can be uniquely extended to a functor $I(f)$ from $I(A)$ to $I(B)$. We arrive in this way at a functor I from **Set** to **Cat**. This functor is right adjoint to the given functor O.

The functor C

Every category $\mathscr{A}$ has an underlying undirected graph, which in turn has connected components. We will call these the **connected components** of $\mathscr{A}$. The class of connected components of $\mathscr{A}$ will be denoted by $C(\mathscr{A})$.

If the category $\mathscr{A}$ is small, then its collections of objects is a set, whence its collection of connected components is a set. Every functor

$$F : \mathscr{A} \longrightarrow \mathscr{B}$$

maps the connected components of $\mathscr{A}$ into connected components of $\mathscr{B}$, and therefore induces a map

$$C(F) : C(\mathscr{A}) \longrightarrow C(\mathscr{B}) .$$

We arrive in this way at a functor C from **Cat** to **Set**. This functor is left adjoint to the previous functor D.

3.3 Historical remarks

Exercise 3.3.1

I asked myself. I could not state any axiomatization of sets, and my principles for working with sets can be described as naive set theory.

[4]The category $I(A)$ may also be constructed by endowing the set A with the preorder $\leq$ with $a \leq a'$ for every two elements a and a' of A, and then regard the preordered set $(A, \leq)$ as a category.

Chapter 4

Representables

4.1 Definitions and examples

Exercise 4.1.26

- We can assign to each topological space X its set of path-components $\pi_0(X)$, resulting in a functor
$$\pi_0 : \mathbf{hTop} \longrightarrow \mathbf{Set}\,.$$
This functor is represented by the one-point topological space.
- We have a functor $(-)^\times$ that assigns to each ring its group of units, i.e., its multiplicative group of invertible elements. When this functor is composed with the forgetful functor from **Grp** to **Set**, it becomes representable by the ring of integral Laurent series $\mathbb{Z}[t, t^{-1}]$.
- Let G be a group. To every G-set X we can assign its set X^G of fixed points, i.e., the set
$$X^G = \{x \in X \mid g \cdot x = x \text{ for every element } g \text{ of } G\}\,.$$
Every homomorphism of G-sets restricts to a map between fixed points, whence the construction $(-)^G$ results in a functor from G-**Set** to **Set**. This functor is represented by the one-point G-set.

Exercise 4.1.27

Let α be a natural isomorphism from H_A to $H_{A'}$. This means that we have for every object B a bijection

$$\alpha_B : \mathscr{A}(B, A) \longrightarrow \mathscr{A}(B, A')\,,$$

such that these bijections are compatible in the following sense: for every morphism

$$g : B \longrightarrow B'$$

in $\mathscr{B}$, the following square diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(B', A) \arrow[r, "\alpha_B"] \arrow[d, "g^*"'] & \mathscr{A}(B', A') \arrow[d, "g^*"] \\
\mathscr{A}(B, A) \arrow[r, "\alpha_{B'}"] & \mathscr{A}(B, A')
\end{tikzcd} \tag{4.1}$$

Under the bijection

$$\alpha_A : \mathscr{A}(A, A) \longrightarrow \mathscr{A}(A, A'),$$

the identity morphism of A has an image in $\mathscr{A}(A, A')$, which we will denote by φ. Similarly, under the bijection

$$\alpha_{A'} : \mathscr{A}(A', A) \longrightarrow \mathscr{A}(A, A),$$

the identity morphism of A' has a unique preimage in $\mathscr{A}(A', A)$, which we will denote by ψ. We have thus found two morphisms

$$\varphi : A \longrightarrow A', \quad \psi : A' \longrightarrow A.$$

We will show in the following that these two morphisms are mutually inverse isomorphisms.

We can consider the commutative diagram (4.1) in the special cases of φ and ψ in place of g. This gives us the following two commutative diagrams:

$$\begin{tikzcd}
\mathscr{A}(A', A) \arrow[r, "\alpha_A"] \arrow[d, "\varphi^*"'] & \mathscr{A}(A', A') \arrow[d, "\varphi^*"] \\
\mathscr{A}(A, A) \arrow[r, "\alpha_{A'}"] & \mathscr{A}(A, A')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A}(A, A) \arrow[r, "\alpha_A"] \arrow[d, "\psi^*"'] & \mathscr{A}(A, A') \arrow[d, "\psi^*"] \\
\mathscr{A}(A', A) \arrow[r, "\alpha_{A'}"] & \mathscr{A}(A', A')
\end{tikzcd}$$

We get from the commutativity of the left-hand diagram that

$$\alpha_{A'}(\psi \circ \varphi) = \alpha_{A'}(\varphi^*(\psi)) = \varphi^*(\alpha_A(\psi)) = \varphi^*(1_{A'}) = 1_{A'} \circ \varphi = \varphi = \alpha_{A'}(1_A),$$

and therefore

$$\psi \circ \varphi = 1_A$$

by the injectivity of $\alpha_{A'}$. We also get from the commutativity of the right-hand diagram that

$$\varphi \circ \psi = \psi^*(\varphi) = \psi^*(\alpha_A(1_A)) = \alpha_{A'}(\psi) = 1_{A'} .$$

These calculations show that the two morphisms φ and ψ are indeed mutually inverse isomorphisms.

The existence of the isomorphisms φ and ψ shows that the two objects A and A' are isomorphic.

Exercise 4.1.28

For every group G, the set $U_p(G)$ may more tactically be expressed at

$$U_p(G) = \{g \in G \mid g^p = 1\} .$$

We see from this alternative description of U_p that every homomorphism of groups

$$\varphi : G \longrightarrow H$$

restricts to a map from $U_p(G)$ to $U_p(H)$. We denote this restriction by $U_p(\varphi)$. This action on morphisms gives us the desired functor U_p from **Grp** to **Set**.

Let G be a group. By the homomorphism theorem, a homomorphism of groups ψ from $\mathbb{Z}/p\mathbb{Z}$ to G corresponds one-to-one to a homomorphism of groups φ from $\mathbb{Z}$ to G with $p\mathbb{Z} \subseteq \ker(\varphi)$, via the formula

$$\psi([n]) = \varphi(n) \qquad \text{for all } n \in \mathbb{Z} .$$

We already know that homomorphisms from $\mathbb{Z}$ to G correspond to elements of G, with the homomorphism φ corresponding to the element $\varphi(1)$ of G. The condition $p\mathbb{Z} \subseteq \ker(\varphi)$ is then equivalent to the condition $p \in \ker(\varphi)$, and thus equivalent to the condition $\varphi(1)^p = 1$. This means altogether that we have a bijection

$$\alpha_G : \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G) \longrightarrow U_p(G) , \quad \psi \longmapsto \psi([1]) .$$

Let us now show that the bijection α_G is natural in G. For this, we need to show that for every homomorphism of groups

$$\varphi : G \longrightarrow H$$

the following diagram commutes:

$$\begin{tikzcd}
\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G) \arrow[r, "\varphi_*"] \arrow[d, "\alpha_G"'] & \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, H) \arrow[d, "\alpha_H"] \\
U_p(G) \arrow[r, "U_p(\varphi)"] & U_p(H)
\end{tikzcd}$$

To this end, we have the chain of equalities

$$\begin{aligned}
\alpha_H(\varphi_*(\psi)) &= \alpha_H(\varphi \circ \psi) \\
&= (\varphi \circ \psi)([1]) \\
&= \varphi(\psi([1])) \\
&= \varphi(\alpha_G(\psi)) \\
&= U_p(\varphi)(\alpha_G(\psi))
\end{aligned}$$

for every element ψ of $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, G)$.

We have overall constructed a natural isomorphism α from the represented functor $\mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$ to the functor U_p. The existence of such an isomorphism shows that the two functors are isomorphic.

Exercise 4.1.29

Let R be a ring. We know from algebra that there exists for every element r of R a unique homomorphism of rings ϕ from $\mathbb{Z}[x]$ to R with $\phi(x) = r$.[1] This means that the map

$$\varepsilon_R : \mathbf{Ring}(\mathbb{Z}[x], R) \longrightarrow R, \quad \phi \longmapsto \phi(x)$$

is bijective. We claim that the bijection ε_R is natural in R.

To prove this naturality, we need to show that for every homomorphism of rings

$$f : R \longrightarrow S,$$

[1] This homomorphism ϕ maps any polynomial $\sum_n a_n x^n$ in $\mathbb{Z}[x]$ to the element $\sum_n a_n r^n$ of R.

the square diagram

$$\begin{tikzcd}
\mathbf{Ring}(\mathbb{Z}[x], R) \arrow[r, "\varepsilon_R"] \arrow[d, "f_*"] & R \arrow[d, "f"] \\
\mathbf{Ring}(\mathbb{Z}[x], S) \arrow[r, "\varepsilon_S"] & S
\end{tikzcd}$$

commutes. To this end, we have for every element ϕ of $\mathbf{Ring}(\mathbb{Z}[x], R)$ the chain of equalities

$$f(\varepsilon_R(\phi)) = f(\phi(x)) = (f \circ \phi)(x) = \varepsilon_R(f \circ \phi) = \varepsilon_R(f_*(\phi)) .$$

This shows that the bijections ε_R, where R ranges through the class of rings, assemble into a natural isomorphism ε from the functor $\mathbf{Ring}(\mathbb{Z}[x], -)$ to the forgetful functor from **CRing** to **Set**.

Exercise 4.1.30

We denote the elements of the Sierpiński space S by 0 and 1, so that the singleton $\{1\}$ is open in S.

Let X be a topological space. We know that we have a bijection

$$\begin{aligned}
\{\text{set-theoretic maps from } X \text{ to } S\} &\longrightarrow \{\text{subsets of } X\} , \\
\chi &\longmapsto \chi^{-1}(1) .
\end{aligned}$$

A map χ from X to S is continuous if and only if for every open subset of S, its preimage under χ is an open subset of X. The preimages $\chi^{-1}(\varnothing) = \varnothing$ and $\chi^{-1}(S) = X$ are open in X independent of the choice of topology on X. Therefore, χ is continuous if and only if the preimage $\chi^{-1}(1)$ is open in X. We thus find that the above bijection restricts to a bijection

$$\begin{aligned}
\{\text{continous maps from } X \text{ to } S\} &\longrightarrow \{\text{open subsets of } X\} , \\
\chi &\longmapsto \chi^{-1}(1) .
\end{aligned}$$

In other words, we have for every topological space X a bijection

$$\alpha_X : \mathbf{Top}(X, S) \longrightarrow \mathcal{O}(X) , \quad \chi \longmapsto \chi^{-1}(1) .$$

Let us check that the bijection α_X is natural in X. For this, let

$$f : X \longrightarrow Y$$

be a continuous map between topological spaces. We need to check that the following diagram commutes:

$$\begin{tikzcd} \mathbf{Top}(X,S) \arrow[r, "\alpha_X"] & \mathcal{O}(X) \\ \mathbf{Top}(Y,S) \arrow[r, "\alpha_Y"] \arrow[u, "f^*"] & \mathcal{O}(Y) \arrow[u, "\mathcal{O}(f)"'] \end{tikzcd}$$

To this end, we have for every element χ of $\mathbf{Top}(Y,S)$ the chain of equalities

$$\begin{aligned} \alpha_X(f^*(\chi)) &= \alpha_X(\chi \circ f) \\ &= (\chi \circ f)^{-1}(1) \\ &= f^{-1}(\chi^{-1}(1)) \\ &= f^{-1}(\alpha_Y(\chi)) \\ &= \mathcal{O}(f)(\alpha_Y(\chi)) \,. \end{aligned}$$

The bijections α_X, where X ranges through the class of topological spaces, therefore assemble into a natural isomorphism α from $\mathbf{Top}(-,X)$ to $\mathcal{O}$.

The existence of such a natural isomorphism shows that the functor $\mathcal{O}$ is represented by the Sierpiński space S.

Exercise 4.1.31

Let $\mathbf{2}$ be the category with two objects, denoted as 0 and 1, and precisely one non-identity morphism, which we denote by i, and which goes from 0 to 1.[2] This category may be depicted as follows:

$$0 \xrightarrow{\quad i \quad} 1$$

Let $\mathscr{A}$ be an arbitrary category. There exists for every morphism

$$f : A \longrightarrow A'$$

[2] In other words, the category $\mathbf{2}$ corresponds to the partially ordered set $\{0,1\}$ with $0 < 1$.

in $\mathscr{A}$ a unique functor F from $\mathbf{2}$ to $\mathscr{A}$ with $F(i) = f$. This functor is given by

$$F(0) = A\,, \quad F(1) = A'\,, \quad F(1_0) = 1_A\,, \quad F(1_1) = 1_{A'}\,, \quad F(i) = f\,.$$

If $\mathscr{A}$ is small, then we have therefore a bijection of sets

$$\varepsilon_{\mathscr{A}} \colon \mathbf{Cat}(\mathbf{2}, \mathscr{A}) \longrightarrow M(\mathscr{A})\,, \quad F \longmapsto F(i)\,.$$

Let us check that this bijection is natural in $\mathscr{A}$. For this, we need to check that for every functor

$$G \colon \mathscr{A} \longrightarrow \mathscr{B}$$

between small categories $\mathscr{A}$ and $\mathscr{B}$ the following square diagram commutes:

$$\begin{tikzcd}
\mathbf{Cat}(\mathbf{2}, \mathscr{A}) \arrow[r, "\varepsilon_{\mathscr{A}}"] \arrow[d, "G_*"'] & M(\mathscr{A}) \arrow[d, "M(G)"] \\
\mathbf{Cat}(\mathbf{2}, \mathscr{B}) \arrow[r, "\varepsilon_{\mathscr{B}}"] & M(\mathscr{B})
\end{tikzcd}$$

To this end, we have for every element F of $\mathbf{Cat}(\mathbf{2}, \mathscr{A})$, the chain of equalities

$$\begin{aligned}
M(G)(\varepsilon_{\mathscr{A}}(F)) &= G(\varepsilon_{\mathscr{A}}(F)) \\
&= G(F(i)) \\
&= (G \circ F)(i) \\
&= \varepsilon_{\mathscr{B}}(G \circ F) \\
&= \varepsilon_{\mathscr{B}}(G_*(F))\,.
\end{aligned}$$

We have thus found that the bijections $\varepsilon_{\mathscr{A}}$, where $\mathscr{A}$ ranges through the class of small categories, assemble into a natural isomorphism ε from the functor $\mathbf{Cat}(\mathbf{2}, -)$ to the functor M. The existence of such a natural isomorphism shows that the functor M is represented by the category $\mathbf{2}$.

Exercise 4.1.32

We know from Exercise 2.1.14 that an adjunction between F and G (with F left adjoint to G) can be characterized as a choice of bijection

$$\Phi_{A,B} \colon \mathscr{A}(A, G(B)) \longrightarrow \mathscr{B}(F(A), B)\,, \quad f \longmapsto \overline{f}$$

where A and B range through the objects of $\mathscr{A}$ and $\mathscr{B}$ respectively, subject to the following condition: for all morphisms

$$p : A' \longrightarrow A\,, \quad q : B \longrightarrow B'$$

in $\mathscr{A}$ and $\mathscr{B}$ respectively, we have the identity

$$\begin{aligned} &\overline{\left(A' \xrightarrow{p} A \xrightarrow{f} G(B) \xrightarrow{G(q)} G(B')\right)} \\ = &\left(F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\overline{f}} B \xrightarrow{q} B'\right). \end{aligned}$$

This identity can equivalently be expressed as the commutativity of the following square diagram:

$$\begin{tikzcd} \mathscr{A}(A, G(B)) \arrow[r, "\Phi_{A,B}"] \arrow[d, "G(q)\circ(-)\circ p"'] & \mathscr{B}(F(A), B) \arrow[d, "q\circ(-)\circ F(p)"] \\ \mathscr{A}(A', G(B')) \arrow[r, "\Phi_{A',B'}"] & \mathscr{B}(F(A'), B') \end{tikzcd}$$

We can now rewrite this commutative diagram in terms of the two functors $\mathscr{A}(-, G(-))$ and $\mathscr{B}(F(-), -)$ as follows:

$$\begin{tikzcd} \mathscr{A}(-, G(-))(A, B) \arrow[r, "\Phi_{A,B}"] \arrow[d, "\mathscr{A}(-,G(-))(p,q)"'] & \mathscr{B}(F(-), -)(A, B) \arrow[d, "\mathscr{B}(F(-),-)(p,q)"] \\ \mathscr{A}(-, G(-))(A', B') \arrow[r, "\Phi_{A',B'}"] & \mathscr{B}(F(-), -)(A', B') \end{tikzcd}$$

That this diagram commutes for any two morphisms p and q as above means precisely that the family $\Phi_{A,B}$, where (A, B) ranges through the objects of the category $\mathscr{A} \times \mathscr{B}$, is a natural transformation from $\mathscr{A}(-, G(-))$ to $\mathscr{B}(F(-), -)$.

We find overall that an adjunction between F and G, with F left adjoint to G, may equivalently be expressed as a natural isomorphism of functors from $\mathscr{A}(-, G(-))$ to $\mathscr{B}(F(-), -)$. This entails that F is left adjoint to G if and only if such a natural isomorphism exists, i.e., if and only if the two functors $\mathscr{A}(-, G(-))$ and $\mathscr{B}(F(-), -)$ are naturally isomorphic.

4.2 The Yoneda lemma

Exercise 4.2.2

Let $\mathscr{A}$ be a locally small category and let X be a functor from $\mathscr{A}$ to **Set**. The map

$$[\mathscr{A}, \mathbf{Set}](\mathrm{H}^A, X) \longrightarrow X(A), \quad \alpha \longmapsto \alpha_A(1_A)$$

is a bijection, which is natural in both A and X.

Exercise 4.2.3

We denote the unique object of M by $\star$. For every functor F from M^{op} to **Set**, we denote by $X(F)$ the resulting right M-set.[3]

(a)

Let H be the unique represented functor from M^{op} to **Set**. This functor is simply $M(-, \star)$ because $\star$ is the unique object of M.

The underlying set of $X(H)$ is given by the set

$$H(\star) = M(\star, \star) = M\,.$$

For every element m of M, the action of m on X is given by the map $H(m)$, whence

$$x \cdot m = H(m)(x) = m^*(x) = xm$$

for every element x of $X(H)$.

This shows overall that $X(H)$ is the right regular M-set.

(b)

We denote the forgetful functor from the category of right M-sets to the category **Set** by U.

For every M-set Y and every element y of $U(Y)$, there exists a unique homomorphism of right M-sets f_y from M to Y with $f_y(1) = y$. This homomorphism is more explicitly given by

$$f_y(m) = ym$$

[3]In other words, we denote by X the isomorphism of categories from the functor category $[M^{\mathrm{op}}, \mathbf{Set}]$ to the category of right M-sets.

for every element m of M. The map

$$\{\text{homomorphisms of right } M\text{-sets } M \to Y\} \longrightarrow U(Y)\,, \quad g \longmapsto g(1)$$

is therefore a bijection.

(c)

Let F be functor from M^{op} to **Set**. We have on the one hand the bijection

$$\left\{ \begin{matrix} \text{natural transformations} \\ H \to F \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{homomorphisms of right } M\text{-sets} \\ X(H) \to X(F) \end{matrix} \right\}, \\ \alpha \longmapsto \alpha_\star\,.$$

We have on the other hand the bijection

$$\left\{ \begin{matrix} \text{homomorphisms of right } M\text{-sets} \\ X(H) \to X(F) \end{matrix} \right\} \longrightarrow F(\star)\,, \\ g \longmapsto g(1)\,,$$

because $X(H)$ is the right regular M-set and because

$$U(X(F)) = F(\star)\,.$$

By combining both of these bijections, we end up with the bijection

$$[M^{\mathrm{op}}, \mathbf{Set}](H, F) \longrightarrow F(\star)\,, \quad \alpha \longmapsto \alpha_\star(1_M) = \alpha_\star(1_\star)\,.$$

The naturality of this bijection can be checked as in the general proof of the Yoneda lemma. Alternatively, one can check that all the above bijections were natural to begin with, and then conclude that the final bijection is also natural.

4.3 Consequences of the Yoneda lemma

Exercise 4.3.15

(a)

If f is an isomorphism in $\mathscr{A}$ then $J(f)$ is an isomorphism in $\mathscr{B}$ because functors preserve isomorphisms.

Suppose on the other hand that $J(f)$ is an isomorphism in $\mathscr{B}$. The morphism f is of the form

$$f : A \longrightarrow A'$$

for objects A and A' of $\mathscr{A}$. It follows that the isomorphism $J(f)$ is of the form

$$J(f) : J(A) \longrightarrow J(A') .$$

There exists by assumption a morphism

$$g' : J(A') \longrightarrow J(A)$$

with both

$$J(f) \circ g' = 1_{J(A')} \quad \text{and} \quad g' \circ J(f) = 1_{J(A)} .$$

There exists a morphism

$$f' : A' \longrightarrow A$$

in $\mathscr{A}$ with $g' = J(f')$ because the functor J is full. We find that

$$J(f' \circ f) = J(f') \circ J(f) = g' \circ J(f) = 1_{J(A)} = J(1_A) ,$$

and therefore $f' \circ f = 1_A$ because the functor J is faithful. We find in the same way that also $f \circ f' = 1_{A'}$. This shows that the morphism f is an isomorphism.

(b)

There exists a unique morphism

$$f : A \longrightarrow A'$$

in $\mathscr{A}$ with $J(f) = g$ because the functor J is full (showing the existence of f) and faithful (showing the uniqueness of f). We find from part (a) of this exercise that the morphism f is an isomorphism because $J(f) = g$ is an isomorphism.

(c)

If the objects A and A' are isomorphic, then so are the objects $J(A)$ and $J(A')$ by the functoriality of J.

Suppose on the other hand that the two objects $J(A)$ and $J(A')$ are isomorphic. This means that there exists an isomorphism g between $J(A)$ and $J(A')$. It follows from part (b) of this exercise that the isomorphism g stems from an isomorphism f between A and A'. The existence of f entails that the objects A and A' are isomorphic.

Exercise 4.3.16

(a)

Let

$$f : A \longrightarrow A'$$

be a morphism in $\mathscr{A}$. The induces natural transformation

$$f_* : \mathrm{H}_A \Longrightarrow \mathrm{H}_{A'}$$

satisfies

$$(f_*)_A(1_A) = f \circ 1_A = f\,.$$

Therefore, the morphism f can be retrieved from its induced natural transformation f_*. This tells us that the functor $\mathrm{H}_\bullet$ is faithful.

(b)

Let A and A' be two objects of $\mathscr{A}$, and let

$$\alpha : \mathrm{H}_A \Longrightarrow \mathrm{H}_{A'}$$

be a natural transformation. Let f be the image of the identity morphism 1_A under the map α_A, i.e., under the map

$$\mathscr{A}(A, A) = \mathrm{H}_A(A) \xrightarrow{\alpha_A} \mathrm{H}_{A'}(A) = \mathscr{A}(A, A')\,.$$

We note that f is a morphism from A to A' in $\mathscr{A}$.

The naturality of α tells us that we have for every morphism

$$g : B \longrightarrow B'$$

in $\mathscr{A}$ the following commutative square diagram:

$$\begin{tikzcd}
\mathrm{H}_A(B') \arrow[r, "\mathrm{H}_A(g)"] \arrow[d, "\alpha_{B'}"'] & \mathrm{H}_A(B) \arrow[d, "\alpha_B"] \\
\mathrm{H}_{A'}(B') \arrow[r, "\mathrm{H}_{A'}(g)"] & \mathrm{H}_{A'}(B)
\end{tikzcd}$$

This diagram may equivalently be written out as follows:

$$\begin{tikzcd}
\mathscr{A}(B', A) \arrow[r, "g^*"] \arrow[d, "\alpha_{B'}"'] & \mathscr{A}(B, A) \arrow[d, "\alpha_B"] \\
\mathscr{A}(B', A') \arrow[r, "g^*"] & \mathscr{A}(B, A')
\end{tikzcd}$$

Let now B be any object of $\mathscr{A}$ and let g be an element of the set $\mathrm{H}_A(B)$. In light of the equality $\mathrm{H}_A(B) = \mathscr{A}(B, A)$, this means that g is a morphism from B to A. We have therefore the following commutative diagram:

$$\begin{tikzcd}
\mathscr{A}(A, A) \arrow[r, "g^*"] \arrow[d, "\alpha_{A}"'] & \mathscr{A}(B, A) \arrow[d, "\alpha_B"] \\
\mathscr{A}(A, A') \arrow[r, "g^*"] & \mathscr{A}(B, A')
\end{tikzcd}$$

The commutativity of this diagram tells us that

$$\alpha_B(g) = \alpha_B(g^*(1_A)) = g^*(\alpha_A(1_A)) = g^*(f) = f \circ g = f_*(g)\,.$$

This shows that the map

$$\alpha_B : \mathrm{H}_A(B) \longrightarrow \mathrm{H}_{A'}(B)$$

is given by f_*.

We have shown that the natural transformations α and f_* from H_A to $\mathrm{H}_{A'}$ coincide in each component, whence they are equal. This shows that each natural transformation from H_A to $\mathrm{H}_{A'}$ stems from a morphism from A to A'. In other words, the functor $\mathrm{H}_\bullet$ is full.

(c)

Suppose that there exists an object A of $\mathscr{A}$ and an element u of $X(A)$ such that the provided universal property holds:

> *For every object B of $\mathscr{A}$ and every element x of $X(B)$, there exists a unique morphism f from B to A in $\mathscr{A}$ such that $x = X(f)(u)$.*

This universal property states that for every object B of $\mathscr{A}$, the map

$$\alpha_B \colon \mathscr{A}(B, A) \longrightarrow X(B), \quad f \longmapsto X(f)(u)$$

is bijective. These bijections are natural. Indeed, let

$$g \colon B \longrightarrow B'$$

be a morphism in $\mathscr{A}$. The resulting diagram

$$\begin{tikzcd}
\mathscr{A}(B', A) \arrow[r, "g^*"] \arrow[d, "\alpha_{B'}"'] & \mathscr{A}(B, A) \arrow[d, "\alpha_B"] \\
X(B') \arrow[r, "X(g)"] & X(B)
\end{tikzcd}$$

commutes, because we have for every element f of its top-left corner the chain of equalities

$$\begin{aligned}
\alpha_B(g^*(f)) &= \alpha_B(f \circ g) \\
&= X(f \circ g)(u) \\
&= (X(g) \circ X(f))(u) \\
&= X(g)(X(f)(u)) \\
&= X(g)(\alpha_{B'}(u)).
\end{aligned}$$

We have thus constructed a natural isomorphism α the functor H_A to the functor X, showing that these functors are isomorphic.

Exercise 4.3.17

Let $\mathscr{A}$ be a discrete category. We begin by describing the presheaf category of $\mathscr{A}$, which we shall denote by $\hat{\mathscr{A}}$.

- A presheaf on $\mathscr{A}$ amounts to a family of sets, indexed by the objects of $\mathscr{A}$.
- Let X and Y be two presheaves on $\mathscr{A}$, and let $(X_A)_A$ and $(Y_A)_A$ be the corresponding families of sets. A natural transformation α from X to Y is simply a family $(\alpha_A)_A$ consisting of a function α_A from X_A to Y_A for every object A of $\mathscr{A}$.

We have provided explicit descriptions of the objects of $\hat{\mathscr{A}}$ and the morphisms between these objects. We see from these explicit descriptions that $\hat{\mathscr{A}}$ is isomorphic to the product category $\prod_{A \in \mathrm{Ob}(\mathscr{A})} \mathbf{Set}$. Moreover, under this isomorphism, the evaluation functor at A corresponds to the projection onto the A-th coordinate.

We now investigate represented presheaves on $\mathscr{A}$ and the Yoneda embedding.

- For every object A of $\mathscr{A}$, the resulting represented presheaf H_A is corresponds to the family of sets $(H_{A'})_{A'}$ for which each set $H_{A'}$ with $A' \neq A$ is empty and the set H_A is only a singleton. In other words,

$$H_{A'} \cong \begin{cases} \{*\} & \text{for } A' = A, \\ \varnothing & \text{otherwise} . \end{cases}$$

- It follows for every other presheaf X on $\mathscr{A}$, that

$$\begin{aligned} \hat{\mathscr{A}}(\mathrm{H}_A, X) &\cong \left(\prod_{A' \in \mathrm{Ob}(\mathscr{A})} \mathbf{Set} \right) \Big((H_{A'})_{A'},\ (X(A'))_{A'} \Big) \\ &\cong \prod_{A' \in \mathrm{Ob}(\mathscr{A})} \mathbf{Set}(H_{A'}, X(A')) \\ &\cong \mathbf{Set}(H_A, X(A)) \times \prod_{\substack{A' \in \mathrm{Ob}(\mathscr{A}) \\ A' \neq A}} \mathbf{Set}(H_{A'}, X(A')) \\ &\cong \mathbf{Set}(\{*\}, X(A)) \times \prod_{\substack{A' \in \mathrm{Ob}(\mathscr{A}) \\ A' \neq A}} \underbrace{\mathbf{Set}(\varnothing, X(A'))}_{\cong \{*\}} \\ &\cong \mathbf{Set}(\{*\}, X(A)) \\ &\cong X(A) . \end{aligned}$$

  This gives us Yoneda's lemma for $\mathscr{A}$.

- We see that for any two distinct objects A and A' of $\mathscr{A}$, there exist no natural transformations between the associated presheaves H_A and $\mathrm{H}_{A'}$, just as there exist no morphisms between A and A'.

  If however $A = A'$, then there exists precise one natural transformation from H_A to $\mathrm{H}_{A'}$, namely the identity natural transformation. Similarly, there exists precisely one morphism from A to A' in $\mathscr{A}$, namely the identity morphism. We see from these observations that the Yoneda embedding of $\mathscr{A}$ into $\hat{\mathscr{A}}$ is full and faithful.

- We see from the above explicit description of represented presheaves that

$$\mathrm{H}_A \cong \mathrm{H}_{A'} \iff A = A' \iff A \cong A'$$

for any two objects A and A' of $\mathscr{A}$.

Exercise 4.3.18

(a)

We first show that the functor $J \circ (-)$ is faithful if the original functor J is faithful. Afterwards, we show that $J \circ (-)$ is full and faithful if the original functor J is full and faithful.

Suppose first the functor J is faithful, and let α and α' be two parallel morphisms in $[\mathscr{B}, \mathscr{C}]$ that have the same image under $J \circ (-)$. In other words, α and α' are two natural transformations of the form

$$\alpha, \alpha' : G \Longrightarrow H$$

for two functors G and H from $\mathscr{B}$ to $\mathscr{C}$, and we have $J\alpha = J\alpha'$. This means that for every object B of $\mathscr{B}$, we have the equalities

$$J(\alpha_B) = (J\alpha)_B = (J\alpha')_B = J(\alpha'_B)\,.$$

It follows that $\alpha_B = \alpha'_B$ for every object B of $\mathscr{B}$ because the functor J is faithful. Therefore, $\alpha = \alpha'$. This shows that the induced functor $J \circ (-)$ is again faithful.

Suppose now that the functor J is both full and faithful. Let G and H be two objects of $[\mathscr{B}, \mathscr{C}]$, i.e., functors from $\mathscr{B}$ to $\mathscr{C}$, and let β be a natural transformation from the resulting functor JG to the resulting functor JH. We claim that there exists a natural transformation α from G to H with $\beta = J\alpha$. (This natural transformation is then unique since J is faithful, as seen above.)

For every object B of $\mathscr{B}$, the component β_B is a morphism from $JG(B)$ to $JH(B)$. There hence exists a unique morphism α_B from $G(B)$ to $H(B)$ with $\beta_B = J(\alpha_B)$ because the functor J is full and faithful. These morphisms α_B, where B ranges through the objects of $\mathscr{B}$, assemble into a transformation α from G to H.

We need to show that the transformation α is natural. We hence need to show that for every morphism

$$g \colon B \longrightarrow B'$$

in $\mathscr{B}$ the following diagram commutes:

$$\begin{tikzcd} G(B) \arrow[r, "G(g)"] \arrow[d, "\alpha_B"'] & G(B') \arrow[d, "\alpha_{B'}"] \\ H(B) \arrow[r, "H(g)"] & H(B') \end{tikzcd}$$

By applying the functor J to this diagram, we get the following diagram:

$$\begin{tikzcd} JG(B) \arrow[r, "JG(g)"] \arrow[d, "\beta_B"'] & JG(B') \arrow[d, "\beta_{B'}"] \\ JH(B) \arrow[r, "JH(g)"] & JH(B') \end{tikzcd}$$

We know that this second diagram commutes by the naturality of β. It follows that the first diagram also commutes because the functor J is faithful. More explicitly, the commutativity of the second diagram tells us that

$$\begin{aligned} J(\alpha_{B'} \circ G(g)) &= J(\alpha_{B'}) \circ JG(g) \\ &= \beta_{B'} \circ JG(g) \\ &= JH(g) \circ \beta_B \\ &= JH(g) \circ J(\alpha_B) \\ &= J(H(g) \circ \alpha_B) \,, \end{aligned}$$

from which it follows that

$$\alpha_{B'} \circ G(g) = H(g) \circ \alpha_B$$

because J is faithful.

We have overall shows that the functor $J \circ (-)$ is again full and faithful.

(b)

The induced functor $J \circ (-)$ from $[\mathscr{B}, \mathscr{C}]$ to $[\mathscr{B}, \mathscr{D}]$ is again full and faithful, whence the assertion follows from Lemma 4.3.8.

(c)

We have the natural isomorphism

$$\mathscr{A}(-, G(-)) \cong \mathscr{B}(F(-), -) \cong \mathscr{A}(-, G'(-))$$

of functors from $\mathscr{A}^{\mathrm{op}} \times \mathscr{B}$ to **Set**. Let

$$H \colon \mathscr{A}^{\mathrm{op}} \longrightarrow [\mathscr{A}, \mathbf{Set}]$$

denote the contravariant Yoneda embedding of $\mathscr{A}$. This functor H is full and faithful, and we might rephrase the natural isomorphism

$$\mathscr{A}(-, G(-)) \cong \mathscr{A}(-, G'(-))$$

as the isomorphism

$$H \circ G \cong H \circ G' .$$

It follows from part (b) of this exercise that $G \cong G'$ because the Yoneda embedding H is full and faithful.

Chapter 5

Limits

5.1 Limits: definition and examples

Exercise 5.1.33

Instead of the direct sum $X_1 \oplus X_2$ we are going use the product $X_1 \times X_2$. Let

$$p_1 : X_1 \times X_2 \longrightarrow X_1 , \quad p_2 : X_1 \times X_2 \longrightarrow X_2$$

be the canonical projection maps. Let Y be another vector space. We know that the two maps p_1 and p_2 induce a bijection

$$\begin{aligned} \{\text{functions } Y \to X_1 \times X_2\} &\longrightarrow \{\text{functions } Y \to X_1\} \times \{\text{functions } Y \to X_2\} , \\ f &\longmapsto (p_1 \circ f, p_2 \circ f) . \end{aligned}$$

(By a "function" we mean a set-theoretic map between the respective underlying sets.) Thanks to the above bijection, it now suffices to show the following statement: a map f from Y to $X_1 \times X_2$ is linear if and only if both its composites $p_1 \circ f$ and $p_2 \circ f$ are linear.

Suppose first that the map f is linear. We observe that the projection maps p_1 and p_2 are both linear. It follows that the composites $p_1 \circ f$ and $p_2 \circ f$ are again linear.

Suppose conversely that both $p_1 \circ f$ and $p_2 \circ f$ are linear. To show that the map f is linear, we need to check that it is both additive and homogeneous.

- We have for every two elements y and y' of Y the chain of equalities

$$\begin{aligned} p_i(f(y + y')) &= (p_i \circ f)(y + y') \\ &= (p_i \circ f)(y) + (p_i \circ f)(y') \\ &= p_i(f(y)) + p_i(f(y')) \\ &= p_i(f(y) + f(y')) \end{aligned}$$

for both $i = 1$ and $i = 2$. This means that the two elements $f(y + y')$ and $f(y) + f(y')$ coincide in both components, and are therefore equal. In other words, we find that

$$f(y + y') = f(y) + f(y') .$$

We have thus shown that the map f is additive.

- That f is homogeneous can be shown in the same way. (More explicitly, let y be a vector of Y and let λ be a scalar. We can conclude for $i = 1, 2$ from the linearity of the maps $p_i \circ f$ and p_i that $p_i(f(\lambda y)) = p_i(\lambda f(y))$. Therefore, $f(\lambda x) = \lambda f(y)$.)

Exercise 5.1.34

If (E, i) is an equalizer of f and g, then the given square diagram is not necessarily a pullback diagram. To see this, we consider in the category **Set** the map $1_{\mathbb{Z}}$ and the map

$$s \colon \mathbb{Z} \longrightarrow \mathbb{Z} , \quad x \longmapsto -x .$$

The equalizer of the two maps $1_{\mathbb{Z}}$ and s is given by the set $\{0\}$ together with the inclusion map i from $\{0\}$ to $\mathbb{Z}$. But the diagram

$$\begin{tikzcd} \{0\} \arrow[r, "i"] \arrow[d, "i"'] & \mathbb{Z} \arrow[d, "s"] \\ \mathbb{Z} \arrow[r, "1"'] & \mathbb{Z} \end{tikzcd}$$

is not a pullback diagram. To see this, we observe that the diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[drr, bend left, "s"] \arrow[ddr, bend right, "1_{\mathbb{Z}}"'] & & \\ & \{0\} \arrow[r, "i"] \arrow[d, "i"'] & \mathbb{Z} \arrow[d, "g"] \\ & \mathbb{Z} \arrow[r, "f"'] & \mathbb{Z} \end{tikzcd}$$

commutes, but that adding the unique morphism from $\mathbb{Z}$ to $\{0\}$ destroys this commutativity.

Suppose conversely that the diagram

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback diagram. We claim that (E, i) is an equalizer of f and g. Indeed, we have $f \circ i = g \circ i$ by the commutativity of the diagram. Let A be an object of $\mathscr{A}$ and let j be a morphism from A to X with $f \circ j = g \circ j$. Then there exists by the universal property of the pullback a unique morphism h from A to E with both $i \circ h = j$ and $i \circ h = j$, i.e., with $i \circ h = j$. This shows that (E, i) is indeed an equalizer of f and g.

Exercise 5.1.35

We denote the ambient category by $\mathscr{A}$, and objects and morphisms in the given diagram as follows:

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "i"'] & F \arrow[r, "l"] \arrow[d, "k"] & Y \arrow[d, "h"] \\ W \arrow[r, "f"] & X \arrow[r, "g"] & Z \end{tikzcd}$$

We have the following chain of equivalences:

$$\begin{aligned} & \text{the left-hand square is a pullback} \\ \iff & \left\{ \begin{array}{l} \text{for every object } A \text{ of } \mathscr{A} \text{ and all morphisms} \\ r \colon A \to W \text{ and } s \colon A \to F \text{ with } f \circ r = k \circ s, \\ \text{there exists a unique morphism } p \text{ from } A \text{ to } E \\ \text{with } i \circ p = r \text{ and } j \circ p = s \end{array} \right. \\ \iff & \left\{ \begin{array}{l} \text{for every object } A \text{ of } \mathscr{A} \text{ and all morphisms} \\ r \colon A \to W,\ t \colon A \to X \text{ and } u \colon A \to Y \\ \text{with } g \circ t = h \circ u \text{ and } f \circ r = t, \\ \text{there exists a unique morphism } p \text{ from } A \text{ to } E \\ \text{with } i \circ p = r \text{ and } k \circ j \circ p = t \text{ and } l \circ j \circ p = u \end{array} \right. \end{aligned} \tag{5.1}$$

$$\iff \left\{\begin{array}{l} \text{for every object } A \text{ of } \mathscr{A} \text{ and all morphisms} \\ r\colon A \to W \text{ and } u\colon A \to Y \text{ with } g \circ f \circ r = h \circ u, \\ \text{there exists a unique morphism } p \text{ from } A \text{ to } E \\ \text{with } i \circ p = r \text{ and } k \circ j \circ p = f \circ r \text{ and } l \circ j \circ p = u \end{array}\right. \tag{5.2}$$

$$\iff \left\{\begin{array}{l} \text{for every object } A \text{ of } \mathscr{A} \text{ and all morphisms} \\ r\colon A \to W \text{ and } u\colon A \to Y \text{ with } g \circ f \circ r = h \circ u, \\ \text{there exists a unique morphism } p \text{ from } A \text{ to } E \\ \text{with } i \circ p = r \text{ and } f \circ i \circ p = f \circ r \text{ and } l \circ j \circ p = u \end{array}\right. \tag{5.3}$$

$$\iff \left\{\begin{array}{l} \text{for every object } A \text{ of } \mathscr{A} \text{ and all morphisms} \\ r\colon A \to W \text{ and } u\colon A \to Y \text{ with } g \circ f \circ r = h \circ u, \\ \text{there exists a unique morphism } p \text{ from } A \text{ to } E \\ \text{with } i \circ p = r \text{ and } l \circ j \circ p = u \end{array}\right.$$

$$\iff \text{the outer rectangle is a pullback}\,.$$

We use for the equivalence (5.1), that the right-hand square is a pullback: instead of running through all morphisms $s\colon A \to F$ we can run through all pairs of morphisms $t\colon A \to X$ and $u\colon A \to Y$ with $g \circ t = h \circ u$. These morphisms s, t and u are then related via $t = k \circ s$ and $u = l \circ s$.

For the equivalence (5.2) we eliminate the explicit mention of the morphism t, since it needs to be given by $f \circ r$. For the equivalence (5.3) we use the commutativity of the left-hand square to replace $k \circ j$ by $f \circ i$ in the last of the four lines.

Exercise 5.1.36

(a)

For every object I of $\mathbf{I}$, let f_I be the composite $p_I \circ h$, and thus also the composite $p_I \circ h'$. We have for every morphism

$$u\colon I \longrightarrow J$$

in $\mathbf{I}$ the equalities

$$D(u) \circ f_I = D(U) \circ p_I \circ h = p_J \circ h = f_J\,.$$

This means that the object A together with the family of morphisms $(f_I)_{I \in \mathrm{Ob}(\mathbf{I})}$ is a cone for the diagram D. It follows from the universal property of the limit L that there exists a unique morphism f from A to L with $p_I \circ f = f_I$ for every object I of $\mathbf{I}$. Both h and h' satisfy this defining property of f, whence we must have $f = h$ and $f = h'$. Therefore, $h = h'$.

(b)

Let now $\mathbf{I}$ be the discrete two-object category. Limits of shape $\mathbf{I}$ are just products. A morphism from $A = 1$ to another set X is the same as an element of X. We arrive therefore at the following result:

> *Let D_1 and D_2 be two sets, and let*
>
> $$p_1 : D_1 \times D_2 \longrightarrow D_1 , \quad p_2 : D_2 \times D_2 \longrightarrow D_2$$
>
> *denote the canonical projections. Two elements x and y of the product $D_1 \times D_2$ are equal if and only if*
>
> $$p_1(x) = p_1(y) \quad \textit{and} \quad p_2(x) = p_2(y) .$$
>
> *In other words, two elements of $D_1 \times D_2$ are equal if and only if they are equal in both coordinates.*

Exercise 5.1.37

The set in question is given by

$$L = \left\{ (x_I)_I \in \prod_{I \in \mathrm{Ob}(\mathbf{I})} D(I) \,\middle|\, \begin{array}{l} D(u)(x_I) = x_J \text{ for every} \\ \text{morphism } u : I \to J \text{ in } \mathbf{I} \end{array} \right\} .$$

For every object I of $\mathbf{I}$ let p_I be the projection onto the I-th component from L to $D(I)$. The elements of L are chosen precisely in such a way that L together with the family of maps $(p_I)_I$ is a cone over the diagram D. It remains to show that the cone $(L, (p_I)_I)$ is universal.

To this end, let $(A, (f_I)_I)$ be another cone over D. We need to show that there exists a unique map f from A to L satisfying the condition $p_I \circ f = f_I$ for every object I of $\mathbf{I}$.

We start by showing the uniqueness of the map f. Given an element a of A, the element $f(a)$ needs to satisfy the equalities

$$p_I(f(a)) = (p_I \circ f)(a) = f_I(a)$$

for every object I of $\mathbf{I}$. This shows that the components of $f(a)$ are uniquely determined by the functions f_I, whence the value $f(a)$ is uniquely determined. This means overall, that the map f is unique.

We now show the existence of the desired map f. We start with the map

$$\tilde{f} : A \longrightarrow \prod_{I \in \mathrm{Ob}(\mathbf{I})} D(I), \quad a \longmapsto (f_I(a))_I .$$

For every element a of A, we have the chain of equalities

$$D(u)(p_I(\tilde{f}(a))) = D(u)(f_I(a)) = f_J(a) = p_J(\tilde{f}(a))$$

for every morphism $u : I \to J$ in $\mathbf{I}$. This tells us that the image of the map $\tilde{f}$ is contained in the subset L of $\prod_{I \in \mathrm{Ob}(\mathbf{I})} D(I)$. We can therefore corestrict the map $\tilde{f}$ to a map

$$f : A \longrightarrow L .$$

The auxiliary map $\tilde{f}$ satisfies the condition $\mathrm{pr}_I \circ \tilde{f} = f_I$ for every object I of $\mathbf{I}$, whence the map f satisfies $p_I \circ f = f_I$ for every object I of $\mathbf{I}$.

Exercise 5.1.38

(a)

Let A be an object of $\mathscr{A}$.

By definition, the object L together with the morphism p is an equalizer for the two morphisms s and t. We have therefore the bijection

$$\begin{aligned} \mathscr{A}(A, L) &\longrightarrow \left\{ g \in \mathscr{A}\left(A, \prod_{I \in \mathrm{Ob}(\mathbf{I})} D(I) \right) \,\middle|\, s \circ g = t \circ g \right\}, \\ f &\longmapsto p \circ f . \end{aligned} \tag{5.4}$$

To better understand the right-hand side of this bijection, we use the bijection

$$\begin{aligned} \mathscr{A}\left(A, \prod_{I \in \mathrm{Ob}(\mathbf{I})} D(I) \right) &\longrightarrow \prod_{I \in \mathrm{Ob}(\mathbf{I})} \mathscr{A}(A, D(I)), \\ g &\longmapsto (\mathrm{pr}_I \circ g)_I . \end{aligned} \tag{5.5}$$

How does the right-hand side of (5.4) look like under this bijection? To answer this question, we need to reformulate the condition $s \circ g = t \circ g$ in terms of the components of g.

To this end, let g be a morphism from A to $\prod_{I \in \mathrm{Ob}(\mathbf{I})} D(I)$. For every object I of $\mathbf{I}$ let g_I be the respective component of g, i.e.,

$$g_I = \mathrm{pr}_I \circ g\,.$$

The two morphisms $s \circ g$ and $t \circ g$ both go from the object A to the product $\prod_{u \colon J \to K \text{ in } \mathbf{I}} D(K)$. These two morphisms hence coincide if and only if they coincide in each component, i.e., if and only if

$$\mathrm{pr}_u \circ s \circ g = \mathrm{pr}_u \circ t \circ g \tag{5.6}$$

for every morphism u in $\mathbf{I}$. The morphisms s and t are defined via the formulae

$$\mathrm{pr}_u \circ s = D(u) \circ \mathrm{pr}_J \quad \text{and} \quad \mathrm{pr}_u \circ t = \mathrm{pr}_K$$

for every morphism $u \colon J \to K$ in $\mathbf{I}$. It follows that (5.6) can be rewritten as

$$D(u) \circ \mathrm{pr}_J \circ g = \mathrm{pr}_K \circ g\,,$$

and therefore as

$$D(u) \circ g_J = g_K$$

for every morphism $u \colon J \to K$ in $\mathbf{I}$.

We find from our discussion that $s \circ g = t \circ g$ if and only if $D(u) \circ g_J = g_K$ for every morphism $u \colon J \to K$ in $\mathbf{I}$. In other words: the bijection (5.5) restricts to a bijection between the right-hand side of (5.4) and

$$\left\{ (g_I)_I \in \prod_{I \in \mathrm{Ob}(\mathbf{I})} \mathscr{A}(A, D(I)) \,\middle|\, \begin{array}{l} D(u) \circ g_J = g_K \text{ for every} \\ \text{morphism } u \colon J \to K \text{ in } \mathbf{I} \end{array} \right\}.$$

By combining these bijections, we arrive at the bijection

$$\begin{aligned} \mathscr{A}(A, L) &\longrightarrow \left\{ (g_I)_I \in \prod_{I \in \mathrm{Ob}(\mathbf{I})} \mathscr{A}(A, D(I)) \,\middle|\, \begin{array}{l} D(u) \circ g_J = g_K \text{ for every} \\ \text{morphism } u \colon J \to K \text{ in } \mathbf{I} \end{array} \right\}, \\ g &\longmapsto (\mathrm{pr}_I \circ p \circ g)_I = (p_I \circ g)_I\,. \end{aligned}$$

This bijection encapsulates that the object L together with the given family of morphisms $(p_I)_I$ is indeed a limit for the given diagram D.

(b)

That $\mathscr{A}$ admits both binary products and a terminal object means precisely that $\mathscr{A}$ admits all finite products. We can therefore form for every finite category $\mathbf{I}$ the products

$$\prod_{I \in \mathrm{Ob}(\mathbf{I})} D(I) \quad \text{and} \quad \prod_{u \colon J \to K \text{ in } \mathbf{I}} D(K) \,.$$

The general construction of limits from part (a) of this exercise can therefore be used without change.

Exercise 5.1.39

Let $\mathscr{A}$ be a category that admits both pullbacks and a terminal object. We denote this terminal object by $*$.

We observe that $\mathscr{A}$ admits binary products, since for any two objects X and Y of A, a product of X and Y is the same as a pullback of the following diagram:

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & * \end{tikzcd}$$

It follows that $\mathscr{A}$ admits all finite products, since it admits binary products and a terminal object.

Next, we will show that the category $\mathscr{A}$ admits equalizers. To this end, we make the following observation:

Proposition 5.A. Let $\mathscr{A}$ be a category admitting binary products, and let

$$f, g \colon X \longrightarrow Y$$

be two parallel morphisms in $\mathscr{A}$. An object E of $\mathscr{A}$ together with a morphism i from E to X is an equalizer of f and g if and only if the following diagram is a pullback diagram:

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "\langle 1_X, g \rangle"] \\ X \arrow[r, "\langle 1_X, f \rangle"'] & X \times Y \end{tikzcd}$$

Proof. We have the following chain of equivalences:

$$
\begin{aligned}
& \text{the above diagram is a pushout diagram} \\
\iff & \left\{ \begin{array}{l} \text{for every object } A \text{ of } \mathscr{A} \text{ and all morphisms} \\ r, s \colon A \to X \text{ with } \langle 1_X, f \rangle \circ r = \langle 1_X, g \rangle \circ s, \\ \text{there exists a unique morphism } h \colon A \to E \\ \text{with } r = i \circ h \text{ and } s = i \circ h \end{array} \right. \\
\iff & \left\{ \begin{array}{l} \text{for every object } A \text{ of } \mathscr{A} \text{ and all morphisms} \\ r, s \colon A \to X \text{ with } \langle r, f \circ r \rangle = \langle s, g \circ s \rangle, \\ \text{there exists a unique morphism } h \colon A \to E \\ \text{with } r = i \circ h \text{ and } s = i \circ h \end{array} \right. \\
\iff & \left\{ \begin{array}{l} \text{for every object } A \text{ of } \mathscr{A} \text{ and all morphisms} \\ r, s \colon A \to X \text{ with } r = s \text{ and } f \circ r = g \circ s, \\ \text{there exists a unique morphism } h \colon A \to E \\ \text{with } r = i \circ h \text{ and } s = i \circ h \end{array} \right. \\
\iff & \left\{ \begin{array}{l} \text{for every object } A \text{ of } \mathscr{A} \text{ and every morphism} \\ t \colon A \to X \text{ with } f \circ t = g \circ t, \\ \text{there exists a unique morphism } h \colon A \to E \\ \text{with } t = i \circ h \end{array} \right. \\
\iff & E \text{ together with } i \text{ is an equalizer for } f \text{ and } g\,,
\end{aligned}
$$

This proves the assertion. ∎

It follows from the above proposition that the category $\mathscr{A}$ admits equalizers, since it admits both binary products and pullbacks. We have overall seen that $\mathscr{A}$ admits finite products and equalizers. It follows from Proposition 5.1.26, part (b) that $\mathscr{A}$ admits all finite limits.

Exercise 5.1.40

We denote objects of **Monic**(A) as pairs (X, m) consisting of an object X of $\mathscr{A}$ and a morphism m from X to A. We denote the class of subobjects of A by $\mathrm{Sub}(A)$.

(a)

Every subset A' of A results in an object of $\mathbf{Monic}(A)$, namely (A', i) where i denotes the inclusion map from A' to A'. Conversely, we can consider for every object (X, m) of $\mathbf{Monic}(A)$ the image of the map m, which is a subset of A'. We have thus found a surjective map

$$I : \operatorname{Ob}(\mathbf{Monic}(A)) \longrightarrow \mathscr{P}(A),$$

assigning to each object (X, m) of $\mathbf{Monic}(A)$ the image of the map m. We shall check in the following that the map I descends to a bijection between $\operatorname{Sub}(A)$ and $\mathscr{P}(A)$. More explicitly, we need to show that two objects of $\mathbf{Monic}(A)$ are isomorphic if and only if they have the same image under I.

Let us first show that isomorphic objects of $\mathbf{Monic}(A)$ have the same image under I. A morphism

$$f : (X, m) \longrightarrow (X', m')$$

in $\mathbf{Monic}(A)$ amounts to the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "m"'] & & X' \arrow[dl, "m'"] \\ & A & \end{tikzcd}$$

It follows from the commutativity of this diagram that the image of m is contained in the image of m', so that $I((X, m)) \subseteq I((X', m'))$. If we view the partially ordered set $\mathscr{P}(A)$ as a category, then this means that the assignment I extends to a functor from $\mathbf{Monic}(A)$ to $\mathscr{P}(A)$. This entails that isomorphic objects of $\mathbf{Monic}(A)$ have the same image under I.

Let us now conversely show that objects of $\mathbf{Monic}(A)$ are isomorphic if they have the same image under I.

Let (X, m) and (X', m') be two objects of $\mathbf{Monic}(A)$ that have the same image under I. This means that the two maps m and m' have the same image in A. Let i be the inclusion map from A' to A. Both m and m' factor through i, in the sense that there exist maps

$$n : X \longrightarrow A', \quad n' : X \longrightarrow A'$$

with $m = i \circ n$ and $m' = i \circ n'$. The maps n and n' are injective because m and m' are injective, but they are also surjective by choice of A'. These maps are hence bijective, and thus invertible.

Let f be the composite $(n')^{-1} \circ n$, which is a map from X to X'. We have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "n"'] \arrow[ddr, bend right, "m"'] & & X' \arrow[dl, "n'"] \arrow[ddl, bend left, "m'"] \\
& A' \arrow[d, "i"] & \\
& A &
\end{tikzcd}$$

The commutativity of the outer part of this diagram tells us that f is a morphism from (X, m) to (X', m') in $\mathbf{Monic}(A)$. The map f is bijective since it is a composite of two bijections. In other words, f is an isomorphism in $\mathbf{Set}$. Let us observe that f is therefore also an isomorphism in $\mathbf{Monic}(A)$.

Proposition 5.B. Let $\mathscr{A}$ be a category and let A be an object of $\mathscr{A}$. Let (X, m) and (X', m') be two objects of $\mathbf{Monic}(A)$ and let f be a morphism from (X, m) to (X', m'). Suppose that f is an isomorphism in $\mathscr{A}$ with inverse f^{-1}.

1. The inverse f^{-1} is a morphism from (X', m') to (X, m) in $\mathbf{Monic}(A)$.
2. The two morphisms f and f^{-1} are mutually inverse in $\mathbf{Monic}(A)$.
3. The morphism f is also an isomorphism in $\mathbf{Monic}(A)$.

Proof. That f is a morphism from (X, m) to (X', m') tells us that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "m"'] & & X' \arrow[dl, "m'"] \\
& A &
\end{tikzcd}$$

1. The commutativity of the above diagram gives us the equality $m = m' \circ f$. Rearranging this equality leads to $m \circ f^{-1} = m'$, which tells us that f^{-1} is a morphism from (X', m') to (X, m).
2. We have in $\mathscr{A}$ the identities

$$f \circ f^{-1} = 1_{X'}\,, \quad f^{-1} \circ f = 1_X\,. \tag{5.7}$$

The composition of morphisms in **Monic**(A) is given by the composition of morphisms in $\mathscr{A}$, and we have $1_X = 1_{(X,m)}$ and $1_{X'} = 1_{(X',m')}$. We can therefore express the identities (5.7) in $\mathscr{A}$ as the identities

$$f \circ f^{-1} = 1_{(X',m')}\,, \quad f^{-1} \circ f = 1_{(X,m)}$$

in **Monic**(A). This tells us that the morphisms f and f^{-1} are also mutually inverse in **Monic**(A).

3. This is a direct consequence of part 2. ∎

We have seen that the morphism f from (X,m) to (X',m') is an isomorphism in $\mathscr{A}$, and therefore also a morphism in **Monic**(A). The existence of this isomorphism shows that the objects (X,m) and (X',m') are isomorphic.

(b)

Subobjects in **Grp** are subgroups, subobjects in **Ring** are subrings and subobjects in $\mathbf{Vect}_{\mathbb{k}}$ are linear subspaces. (This can be proven with the same argumentation as for **Set** in part (a) of this exercise.)

(c)

We know that the monomorphisms in **Top** are precisely those continuous maps that are injective. A subobject of a topological space X in the categorical sense is therefore a subset Y of X together with a topology on Y for which the inclusion map from Y to X is continuous. In other words, the topology on Y needs to be coarser than the subspace topology induced from X.

We find in particular that X typically admits more subobjects than just its subspaces.

Exercise 5.1.41

We denote the ambient category by $\mathscr{A}$. We find that

$$\begin{aligned} &\text{the given diagram is a pullback diagram} \\ \iff &\left\{ \begin{array}{l} \text{for every object } A \text{ of } \mathscr{A} \text{ and} \\ \text{all morphisms } r, s \colon A \to X \text{ with } f \circ r = f \circ s, \\ \text{there exists a unique morphism } t \colon A \to X \\ \text{with } r = 1_A \circ t \text{ and } s = 1_A \circ t \end{array} \right. \end{aligned}$$

$$\iff \begin{cases} \text{for every object } A \text{ of } \mathscr{A} \text{ and} \\ \text{all morphisms } r, s\colon\ A \to X \text{ with } f \circ r = f \circ s, \\ \text{there exists a unique morphism } t\colon\ A \to X \\ \text{with } r = t \text{ and } s = t \end{cases}$$

$$\iff \begin{cases} \text{for every object } A \text{ of } \mathscr{A} \text{ and} \\ \text{all morphisms } r, s\colon\ A \to X \text{ with } f \circ r = f \circ s, \\ \text{we have } r = s \end{cases}$$

$$\iff f \text{ is a monomorphism}\,.$$

Exercise 5.1.42

We need to show that for every two parallel morphisms

$$r, s\colon\ Y \longrightarrow X'$$

in $\mathscr{A}$ with $m' \circ r = m' \circ s$, we have $r = s$. We find that

$$m \circ f' \circ r = f \circ m' \circ r = f \circ m' \circ s = m \circ f' \circ s$$

by the commutativity of the given diagram. It follows that

$$f' \circ r = f' \circ s$$

because the morphism m is monic. We know from the universal property of the pullback that any morphism t from Y to X' is uniquely determined by its pair of composites $m' \circ t$ and $f' \circ t$. We have seen that both $m' \circ r = m' \circ s$ and $f' \circ r = f' \circ s$, so we conclude that $r = s$.

5.2 Colimits: definition and examples

Exercise 5.2.21

We denote the ambient category by $\mathscr{A}$.

Proposition 5.C. Let $\mathscr{A}$ be a category, let

$$s, t\colon\ A \longrightarrow B$$

be two parallel morphisms in $\mathscr{A}$ and let (E, i) be an equalizer of s and t. Then i is a monomorphism.

Proof. It follows from Exercise 5.1.36, part (a) that a morphism f in $\mathscr{A}$ with codomain A is uniquely determined by its composite $i \circ f$. ∎

If $s = t$, then X together with the morphism 1_X is an equalizer of s and t, which shows in particular that an equalizer of s and t exists. We also have for every equalizer (E, i) of s and t the chain of equivalences

$$
\begin{aligned}
& i \text{ is an isomorphism} \\
\iff & i_* : \mathscr{A}(-, E) \longrightarrow \mathscr{A}(-, X) \text{ is a natural isomorphism} \\
\iff & \begin{cases} \text{for every object } A \text{ of } \mathscr{A}, \\ \text{the map } i_* : \mathscr{A}(A, E) \to \mathscr{A}(A, X) \text{ is bijective} \end{cases} \\
\iff & \begin{cases} \text{for every object } A \text{ of } \mathscr{A}, \\ \text{the map } i_* : \mathscr{A}(A, E) \to \mathscr{A}(A, X) \text{ is surjective} \end{cases} \qquad (5.8) \\
\iff & \begin{cases} \text{for every object } A \text{ of } \mathscr{A} \text{ and} \\ \text{every morphism } f : A \to X, \\ \text{there exists a morphism } g : A \to E \\ \text{with } f = i \circ g \end{cases} \\
\iff & \begin{cases} \text{for every object } A \text{ of } \mathscr{A} \text{ and} \\ \text{every morphism } f : A \to X, \\ \text{we have } s \circ f = t \circ f \end{cases} \qquad (5.9) \\
\iff & s = t \,.
\end{aligned}
$$

For the equivalence (5.8) we use that the equalizer i is a monomorphism. For the equivalence (5.9) we use that, according to the universal property of the equalizer i, a morphism f factors through i if and only if $s \circ f = t \circ f$.

This shows the statement about equalizers. The statement about coequalizers follows from the following chain of equivalences:

$$
\begin{aligned}
& s = t \text{ in } \mathscr{A} \\
\iff & s^{\mathrm{op}} = t^{\mathrm{op}} \text{ in } \mathscr{A}^{\mathrm{op}} \\
\iff & \begin{cases} s^{\mathrm{op}} \text{ and } t^{\mathrm{op}} \text{ admit an equalizer in } \mathscr{A}^{\mathrm{op}}, \\ \text{which is furthermore an isomorphism} \end{cases} \\
\iff & \begin{cases} s \text{ and } t \text{ admit a coequalizer in } \mathscr{A}, \\ \text{which is furthermore an isomorphism} \end{cases}
\end{aligned}
$$

Exercise 5.2.22

(a)

The coequalizer of f and 1_X is given by the quotient set $X/{\sim}$, where $\sim$ is the equivalence relation on X generated by

$$x \sim f(x) \qquad \text{for every } x \in X .$$

This equivalence relation can also be described more explicitly as follows: two elements x and y of X are equivalent with respect to $\sim$ if and only if they are contained in the same f-orbit, i.e., if and only if there exist natural exponents n and m with

$$f^n(x) = f^m(y) .$$

Alternatively, let Γ be the following directed graph: the vertices of Γ are the elements of X, and there exists an edge from x to y in Γ if and only if $y = f(x)$. Two elements of X are equivalent with respect to $\sim$ if and only if they lie in the same undirected connected component of Γ. The set $X/{\sim}$ can be identified with the set of undirected connected components of Γ.

We want to give special attention to the case that f is bijective: in this case, the elements of X that are equivalent to a specific element x are precisely those of the form $f^n(x)$ with $n \in \mathbb{Z}$.

(b)

Let X be a topological space and let

$$f : X \longrightarrow X$$

be a continuous map. To construct the coequalizer of f we take the construction from part (a) of this exercise and endow the quotient $X/{\sim}$ with the quotient topology induced from X.

For $X = \mathbb{S}^1$ we may consider the rotation map

$$f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1 , \quad x \longmapsto \mathrm{e}^{2\pi \mathrm{i} \alpha} x$$

for some irrational number α. For every element of X, its orbit under f is a countable, dense subset of $\mathbb{S}^1$. The quotient space $X/{\sim}$ is therefore an uncountable, indiscrete topological space.

Exercise 5.2.23

(a)

The inclusion homomorphism i from $\mathbb{N}$ to $\mathbb{Z}$ is not surjective because the element -1 of $\mathbb{Z}$ is not contained in its image.

Let M be a monoid and let f be a homomorphism of monoids from $(\mathbb{Z}, +, 0)$ to M. Every natural number n is invertible in $(\mathbb{Z}, +, 0)$, with its inverse given by the non-positive integer $-n$. It follows that the value $f(n)$ is invertible in M, with inverse given by $f(-n)$. The value $f(-n)$ as therefore uniquely determined by the value $f(n)$.

This shows that the homomorphism f is uniquely determined by its composite $f \circ i$. This in turn shows that i is an epimorphism in the category of monoids.

(b)

The inclusion homomorphism i from $\mathbb{Z}$ to $\mathbb{Q}$ is not surjective because the element $1/2$ of $\mathbb{Q}$ is not contained in its image.

Let R be a ring and let f be a homomorphism of rings from $\mathbb{Q}$ to R. Every non-zero element n of $\mathbb{Z}$ is invertible in $\mathbb{Q}$, whence the element $f(n)$ is invertible in R. Every element x of $\mathbb{Q}$ is of the form $x = p/q$ for some integers p and q with q non-zero. It follows for the element x that

$$f(x) = f\left(\frac{p}{q}\right) = f(p \cdot q^{-1}) = f(p)f(q)^{-1} .$$

This shows that the homomorphism f is uniquely determined by the values $f(n)$ with $n \in \mathbb{Z}$, and is therefore uniquely determined by its composite $f \circ i$. This shows that i is an epimorphism.

Exercise 5.2.24

(a)

Recall 5.D. For every map of the form

$$e \colon A \longrightarrow X ,$$

the equivalence relation $\sim$ on A induced by e is given by

$$a \sim a' \iff e(a) = e(a') \qquad \text{for all } a, a' \in A .$$

We denote the class of quotient objects of A by $\operatorname{Quot}(A)$, and the set of equivalence relations on A by $\operatorname{Equiv}(A)$. We denote the objects of $\mathbf{Epic}(A)$ as pairs (X, e), consisting of an object X and an epimorphism e from A to X. Given such an object, we refer to the equivalence relation induces by e on A as the equivalence relation induced by (X, e).

We have a map

$$E\colon\ \operatorname{Ob}(\mathbf{Epic}(A)) \longrightarrow \operatorname{Equiv}(A)$$

that assigns to each object (X, e) of $\mathbf{Epic}(A)$ the equivalence relation induced by e. We shall show in the following that the map E descends to a bijection from $\operatorname{Quot}(A)$ to $\operatorname{Equiv}(A)$. For this, we need to show that the map E is surjective, and that two objects of $\mathbf{Epic}(A)$ induce the same equivalence relation on A if and only if they are isomorphic.

We can consider for every equivalence relation $\sim$ on A the quotient set $A/{\sim}$ and the canonical quotient map p from A to $A/{\sim}$. The map p is surjective, whence $(A/{\sim}, p)$ is an object of $\mathbf{Epic}(A)$. This object induces the equivalence relation $\sim$, and is therefore a preimage for $\sim$ with respect to E. This shows that the map E is surjective.

Let us now show that isomorphic objects of $\mathbf{Epic}(A)$ have the same image under E.

Let (X, e) and (X', e') be two objects in $\mathbf{Epic}(A)$, and suppose that there exists a morphism f from (X, e) to (X', e') in $\mathbf{Epic}(A)$. This means that f is a map from X to X' that makes the following diagram commute:

$$\begin{tikzcd} & A \arrow[dl, "e"'] \arrow[dr, "e'"] & \\ X \arrow[rr, "f"] & & X' \end{tikzcd}$$

It follows for any two elements a and a' of A, that

$$e(a) = e(a') \implies f(e(a)) = f(e(a')) \implies e'(a) = e'(a').$$

This means that the equivalence relation induced by e implies the equivalence relation induced by e'.

If (X, e) and (X', e') are isomorphic, then there exist morphisms between them in both directions. It then follows that both e and e' induce the same equivalence relation on A.

Let us now show that objects of $\mathbf{Epic}(A)$ that induce the same equivalence relation are already isomorphic. To do so, let (X, e) be an object of $\mathbf{Epic}(A)$ and let $\sim$ be the equivalence relation induced by (X, e). Let furthermore p be the canonical quotient map from A to $A/{\sim}$. We show in the following that

$$(X, e) \cong (A/{\sim}, p) .$$

This then entails that the object (X, e) is determined by the equivalence relation $\sim$ up to isomorphism.

That (X, e) is an object of $\mathbf{Epic}(A)$ means that X is a set and e is an epimorphism from A to X. More specifically, e is an epimorphism in $\mathbf{Set}$, and thus a surjective map. It follows that the map e factors through a bijection

$$f : A/{\sim} \longrightarrow X , \quad [a] \longmapsto e(a) .$$

This induced bijection makes the diagram

$$\begin{tikzcd} & A \arrow[dl, "p"'] \arrow[dr, "e"] & \\ A/{\sim} \arrow[rr, "f"] & & X \end{tikzcd}$$

commute, which tells us that f is a morphism from $(A/{\sim}, p_{\sim})$ to (X, e) in the category $\mathbf{Epic}(A)$.

We make the following observation:

Proposition 5.E. Let $\mathscr{A}$ be a category and let A be an object of $\mathscr{A}$. Let (X, e) and (X', e') be two objects of $\mathbf{Epic}(A)$ and let f be a morphism from (X, e) to (X', e'). Suppose that f is an isomorphism in $\mathscr{A}$ with inverse f^{-1}.

1. The inverse f^{-1} is a morphism from (X', m') to (X, m) in $\mathbf{Epic}(A)$.
2. The two morphisms f and f^{-1} are mutually inverse in $\mathbf{Epic}(A)$.
3. The morphism f is also an isomorphism in $\mathbf{Epic}(A)$.

Proof. The given proposition is the dual of Proposition 5.B (page 131). ∎

It follows from the above proposition that the morphism f from $(A/{\sim}, p)$ to (X, e) is an isomorphism. The two objects $(A/{\sim}, p)$ and (X, e) are therefore isomorphic.

(b)

Given a group G and object (X, e) of $\mathbf{Epic}(G)$, the kernel of e is a normal subgroup of G. We can adapt the argumentation from part (a) of this exercise to show that this assignment yields a bijection between $\mathrm{Quot}(G)$ and the set of normal subgroups of G.

Exercise 5.2.25

(a)

We denote the ambient category by $\mathscr{A}$.

split monomorphism $\implies$ regular monomorphism

Let

$$m \colon A \longrightarrow B$$

be a split monomorphism in $\mathscr{A}$. This means that there exists a morphism

$$e \colon B \longrightarrow A$$

such that $e \circ m = 1_A$.

We note that the morphism m is a monomorphism: for every object X of $\mathscr{A}$ and any two parallel morphisms

$$f, g \colon X \longrightarrow A$$

with $m \circ f = m \circ g$, we have

$$f = e \circ m \circ f = e \circ m \circ g = g\,.$$

We show now that m is an equalizer of the two morphisms

$$m \circ e, 1_B \colon B \longrightarrow B\,.$$

We have the equalities

$$(m \circ e) \circ m = m \circ e \circ m = m \circ 1_A = m = 1_B \circ m\,,$$

which shows that the diagram

$$A \xrightarrow{m} B \underset{1_B}{\overset{m \circ e}{\rightrightarrows}} B$$

is a fork. To see that this fork is universal, let X be an object of $\mathscr{A}$ and let f be a morphism from X to B such that

$$(m \circ e) \circ f = 1_B \circ f\,.$$

This means that $m \circ e \circ f = f$. It follows for the composite $f' \coloneqq e \circ f$, which is a morphism from X to A, that

$$m \circ f' = m \circ e \circ f = f\,.$$

This shows that the morphism f factors through m. We have already shown that m is a monomorphism, whence this factorization is unique.

regular monomorphism $\implies$ monomorphism

Let (E, i) be an equalizer of two parallel morphisms

$$s, t :\ X \longrightarrow Y$$

in $\mathscr{A}$. Let A be an object of $\mathscr{A}$ and let

$$f, g :\ A \longrightarrow E$$

be two morphisms with $i \circ f = i \circ g$. We denote this common composite by r. We have

$$s \circ r = s \circ i \circ f = t \circ i \circ g = t \circ r\,,$$

whence there exists by the universal property of the equalizer (E, i) a unique morphism h from A to E with $r = i \circ h$. Both f and g satisfy the role of h, whence $h = f$ and $h = g$ by the uniqueness of h, and thus $f = g$.

(b)

The inclusion homomorphism m from $2\mathbb{Z}$ to $\mathbb{Z}$ is injective, and therefore a monomorphism in **Ab**. But it is not a split monomorphism in **Ab**. There would otherwise exist a homomorphism e from $\mathbb{Z}$ to $2\mathbb{Z}$ with $e \circ m = 1_{2\mathbb{Z}}$. This would entail for the element $x \coloneqq e(1)$ that

$$2x = 2e(1) = e(2) = e(m(2)) = 2\,.$$

But no element x of $2\mathbb{Z}$ has this property.

Let now

$$m:\ A \longrightarrow B$$

be an arbitrary monomorphism in **Ab**. Let B' be the image of m and let p be the canonical quotient map from B to B/B'. The morphism m is the equalizer of p and the zero homomorphism. It is therefore a regular monomorphism.

(c)

Let

$$s, t:\ X \longrightarrow Y$$

be two parallel morphisms in **Top** and let (E, i) be an equalizer of r and s. We already know that i is a monomorphism, and therefore injective. We show in the following that E carries the subspace topology induced from X. To prove this, we show that for every topological space A and every map f from A to E, the map f is continuous if and only if the composite $i \circ f$ is continuous.

We know that the map i is continuous (since it is a morphism in **Top**), so if f is continuous, then so is the composite $i \circ f$.

Suppose conversely that the composite $i \circ f$ is continuous. Let us abbreviate this composite by g. The map g is a morphism in **Top** that satisfies

$$s \circ g = s \circ i \circ f = t \circ i \circ f = t \circ g\,.$$

It follows from the universal property of the equalizer (E, i) that there exists a unique continuous map f' from A to E with $g = i \circ f'$. The two maps f and f' satisfy the equalities

$$i \circ f = g = i \circ f'\,,$$

and we already know that i is a monomorphism. It follows that $f = f'$, which tells us in particular that the map f is continuous, since f' is continuous.

We have not characterized monomorphisms and regular monomorphisms in **Top**. A morphism in **Top** is a monomorphism if and only if it is injective. A monomorphism is regular if and only if the topology of its domain is the subspace topology induces from its codomain.

We can regard any set X either as an indiscrete topological space $I(X)$ or as a discrete topological space $D(X)$. The identity map from $I(X)$ to $D(X)$ is bijective, therefore injective, and thus a monomorphism. But it is not a regular monomorphism as long as X contains at least two distinct elements, because $I(X)$ does not carry the subspace topology induced from $D(X)$.

Exercise 5.2.26

Dualizing part (a) of Exercise 5.2.25 gives us the implications

$$\text{split epimorphism} \implies \text{regular epimorphism} \implies \text{epimorphism}\,.$$

(a)

An isomorphism is both a split monomorphism and a split epimorphism, and therefore both a monomorphism and a regular epimorphism.

Suppose now conversely that a morphism

$$f \colon A \longrightarrow B$$

is both a monomorphism and a regular epimorphism. There exist by assumption two parallel morphisms

$$s, t \colon C \longrightarrow A$$

such that f is the coequalizer of s and t. This entails that $f \circ s = f \circ t$. It follows that $s = t$ because f is a monomorphism. It further follows from Exercise 5.2.21 that f is an isomorphism.

(b)

We are only left to show the implication

$$\text{epimorphism} \implies \text{split epimorphism}\,,$$

which is provided by the axiom of choice.

(c)

If a category $\mathscr{A}$ satisfies the axiom of choice, then every morphism in $\mathscr{A}$ that is both a monomorphism and an epimorphism is already an isomorphism: it is both a monomorphism and a split epimorphism, therefore both a monomorphism and a regular epimorphism, and therefore an isomorphism by part (a) of this exercise.

Therefore, if **Top** were to satisfy the axiom of choice, then every bijective continuous map would already be a homeomorphism. But we know that this is not the case. We thus see that **Top** does not satisfy the axiom of choice.

The homomorphism of groups

$$p : \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}, \quad x \longmapsto [x]$$

is an epimorphism in **Grp** that does not admit a right-inverse. It is therefore an epimorphism, but not a split epimorphism. The existence of this example shows that **Grp** does not satisfy the axiom of choice.

Exercise 5.2.27

We consider pullbacks, pushouts, and composition.

Monomorphisms, pullbacks

We have seen in Exercise 5.1.42 that the class of monomorphisms is stable under pullbacks.

Monomorphisms, pushouts

The class of monomorphisms is not necessarily stable under pushouts.

To construct a counterexample we consider the category **CRing** of commutative rings. The coproduct of two commutative rings R and S in **CRing** is their tensor product $R \otimes_{\mathbb{Z}} S$, the canonical homomorphism from R and S into their coproduct are given by the mappings $r \mapsto r \otimes 1$ and $s \mapsto 1 \otimes s$. The initial object of **CRing** is given by the ring of integers, $\mathbb{Z}$. The following diagram is therefore a pushout diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & R \arrow[d] \\ S \arrow[r] & R \otimes_{\mathbb{Z}} S \end{tikzcd}$$

We may consider for R and S the two rings $\mathbb{Q}$ and $\mathbb{Z}/2$. We have

$$\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/2) = 0,$$

and therefore the following pushout diagram:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] \\
\mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

The inclusion homomorphism from $\mathbb{Z}$ to $\mathbb{Q}$ is injective, and therefore a monomorphism. But the homomorphism from $\mathbb{Z}/2$ to 0 is not injective, and therefore not a monomorphism.

Monomorphisms, composition

The class of monomorphisms is closed under composition. To see this, let

$$m : A \longrightarrow B, \quad m' : B \longrightarrow C$$

be two composable monomorphisms in some category $\mathscr{A}$. Let

$$f, g : X \longrightarrow A$$

be two morphisms $\mathscr{A}$ with $m' \circ m \circ f = m' \circ m \circ g$. Then $m \circ f = m \circ g$ because m' is a monomorphism, and thus $f = g$ because m is a monomorphism. This shows that $m' \circ m$ is again a monomorphism.

Regular monomorphisms, pullbacks

The class of regular monomorphisms is closed under pullbacks.

Let (E, i) be an equalizer of two morphisms

$$s, t : A \longrightarrow B$$

is some category $\mathscr{B}$. Suppose that i is part of a pullback diagram of the following form:

$$\begin{tikzcd}
E' \arrow[r, "g"] \arrow[d, "i'"'] & E \arrow[d, "i"] \\
A' \arrow[r, "f"'] & A
\end{tikzcd}$$

Let s' and t' be the composites

$$s' := s \circ f\,, \quad t' := t \circ f\,.$$

We may extend the above diagram as follows:

$$\begin{tikzcd}
E' \arrow[r, "g"] \arrow[d, "i'"'] & E \arrow[d, "i"] \\
A' \arrow[r, "f"] \arrow[dr, shift left, "t'"] \arrow[dr, shift right, "s'"'] & A \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
 & B
\end{tikzcd}$$

We claim that (E', i') is an equalizer of s' and t'.

To prove this, let

$$h : C \longrightarrow A'$$

be a morphism in $\mathscr{A}$ with $s' \circ h = t' \circ h$. We need to show that there exists a unique morphism k from C to E' with $h = i' \circ k$. We already know that i' is again a monomorphism, so it only remains to show the existence of k.

We have by assumption the equalities

$$s \circ f \circ h = s' \circ h = t' \circ h = t \circ f \circ h\,.$$

It follows from the universal property of the equalizer (E, i) that there exists a unique morphism k' from C to E with

$$i \circ k' = f \circ h\,.$$

It further follows from the universal property of the pullback (E', g, i') that there exist a unique morphism

$$k : C \longrightarrow E'$$

with both $g \circ k = k'$ and $i' \circ k = h$. This proves the desired existence of k.

The above argument results in the following diagram:

$$\begin{tikzcd}
C \arrow[dr, "k"] \arrow[drr, bend left, "k'"] \arrow[ddr, bend right, "h"'] & & \\
& E' \arrow[r, "g"] \arrow[d, "i'"'] & E \arrow[d, "i"] \\
& A' \arrow[r, "f"] \arrow[dr, shift left, "t'"] \arrow[dr, shift right, "s'"'] & A \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
& & B
\end{tikzcd}$$

Regular monomorphisms, pushouts

According to [MSE13a], the class of regular monomorphisms is not necessarily closed under pushouts.

Regular monomorphisms, composition

According to [AHS06, Exercise 7J], the class of regular monomorphisms is not necessarily closed under composition.

Split monomorphisms, pullbacks

The class of split monomorphisms is not necessarily closed under pullbacks.

To see this, we consider the category **Set**. We observe that monomorphism in **Set** are split-monomorphisms, except for those whose domain is empty, but codomain is non-empty. Let X be a set and let A and B be two subsets of X. Then the diagram

$$\begin{tikzcd}
A \cap B \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & X
\end{tikzcd}$$

is a pullback diagram, where each morphism is the respective inclusion map. If A and B are non-empty and disjoint, then it follows that the inclusion map from A to X is a split monomorphism, but the inclusion map from $A \cap B = \emptyset$ to B is not.

Split monomorphisms, pushouts

The class of split monomorphisms is closed under pushouts.

To see this, let

$$m: A \longrightarrow B$$

be a split monomorphism, and suppose that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "m"'] & A' \arrow[d, "m'"] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$

is a pushout diagram. There exists by assumption a morphism

$$e: B \longrightarrow A$$

with $e \circ m = 1_A$. The two morphisms

$$1_{A'}: A' \longrightarrow A', \quad f \circ e: B \longrightarrow A'$$

satisfy

$$(f \circ e) \circ m = f \circ e \circ m = f \circ 1_A = f = 1_{A'} \circ f .$$

By the universal property of the pushout, there hence exists a unique morphism e' from B' to A' with both $e' \circ m' = 1_{A'}$ and $e' \circ g = f \circ e$. This shows that the morphism m' is again a split monomorphism.

Split monomorphisms, composition

The class of split monomorphisms is closed under composition.

Let

$$m: A \longrightarrow B, \quad m': B \longrightarrow C$$

be two composable split monomorphisms. There exist by assumption morphisms

$$e: B \longrightarrow A, \quad e': C \longrightarrow B$$

with $e \circ m = 1_A$ and $e' \circ m' = 1_B$. It follows that

$$(e \circ e') \circ (m' \circ m) = e \circ e' \circ m' \circ m = e \circ 1_B \circ m = e \circ m = 1_A \,.$$

This shows that the composite $m' \circ m$ is again a split monomorphism.

Epimorphisms, pullbacks

We have seen that the class of monomorphisms is not necessarily closed under pushouts. It follows by duality that the class of epimorphisms is not necessarily closed under pullbacks.

Let us also give an explicit example: given a topological space X and two subspaces A and B of X, we have the following pullback diagram, in which the arrows are the respective inclusion maps:

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

The same holds true if we don't consider the entire category **Top**, but instead only the full subcategory of Hausdorff spaces. In this category, we have therefore the following pullback diagram:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] \\ \mathbb{R} \setminus \mathbb{Q} \arrow[r] & \mathbb{R} \end{tikzcd}$$

The inclusion map $\mathbb{Q} \to \mathbb{R}$ is an epimorphisms in this category, whereas the inclusion map $\varnothing \to \mathbb{R} \setminus \mathbb{Q}$ is not.

Epimorphisms, pushouts

We have seen that the class of monomorphisms is closed under pullbacks. It follows by duality that the class of epimorphisms is closed under pushouts.

Epimorphisms, composition

We have seen that the class of monomorphisms is closed under composition. It follows by duality that the class of epimorphisms is again closed under composition.

Regular epimorphisms, pullbacks

We have seen that the class of regular monomorphisms in not necessarily closed under pushouts. It follows by duality that the class of regular epimorphisms is not necessarily closed under pullbacks.

Regular epimorphisms, pushouts

We have seen that the class of regular monomorphisms is closed under pullbacks. It follows by duality that the class of regular epimorphisms is closed under pushouts.

Regular epimorphisms, composition

We have seen that the class of regular monomorphisms is not necessarily closed under composition. It follows by duality that the class of regular epimorphisms is not necessarily closed under composition.

Split epimorphisms, pullbacks

We have seen that the class of split monomorphisms is closed under pushouts. It follows by duality that the class of split epimorphisms is closed under pullback.

Split epimorphisms, pushouts

We have seen that the class of split monomorphisms is not necessarily closed under pullbacks. It follows by duality that the class of split monomorphisms is not necessarily closed under pushouts.

Split epimorphisms, composition

We have seen that the class of split monomorphisms is closed under composition. It follows by duality that the class of split epimorphisms is also closed under composition.

5.3 Interactions between functors and limits

Exercise 5.3.8

We denote the desired functor by P. We need no define the action of P on morphism and then check the functoriality of this action.

Let

$$(f, g) : (X, Y) \longrightarrow (X', Y')$$

be a morphism in $\mathscr{A} \times \mathscr{A}$. There exists by the universal property of the product $X' \times Y'$ a unique morphism $f \times g$ from $X \times Y$ to $X' \times Y'$ that makes the following diagram commute:

$$\begin{tikzcd}
 & X \times Y \arrow[dl, "p_1^{X,Y}"'] \arrow[dr, "p_2^{X,Y}"] \arrow[dd, dashed, "f \times g"] & \\
X \arrow[dd, "f"'] & & Y \arrow[dd, "g"] \\
 & X' \times Y' \arrow[dl, "p_1^{X',Y'}"'] \arrow[dr, "p_2^{X',Y'}"] & \\
X' & & Y'
\end{tikzcd}$$

We let $P(f, g)$ be this morphism $f \times g$.

We have for every object (X, Y) of $\mathscr{A} \times \mathscr{A}$ the following commutative diagram:

$$\begin{tikzcd}
 & X \times Y \arrow[dl, "p_1^{X,Y}"'] \arrow[dd, "1_X \times 1_Y"] \arrow[dr, "p_2^{X,Y}"] & \\
X \arrow[dd, "1_X"'] & & Y \arrow[dd, "1_Y"] \\
 & X \times Y \arrow[dl, "p_1^{X,Y}"'] \arrow[dr, "p_2^{X,Y}"] & \\
X & & Y
\end{tikzcd}$$

The commutativity of this diagram tells us that the morphism $1_{X\times Y}$ satisfies the defining property of the morphism $1_X \times 1_Y$, whence

$$1_X \times 1_Y = 1_{X\times Y} .$$

We have therefore the chain of equalities

$$P(1_{(X,Y)}) = P(1_X, 1_Y) = 1_X \times 1_Y = 1_{X\times Y} = 1_{P(X,Y)} .$$

Let now

$$(f, g) : (X, Y) \longrightarrow (X', Y') , \quad (f', g') : (X', Y') \longrightarrow (X'', Y'')$$

be two composable morphisms in $\mathscr{A}\times\mathscr{A}$. We have the following commutative diagram:

$$\begin{tikzcd}
 & X \times Y \arrow[dl, "p_1^{X,Y}"'] \arrow[dr, "p_2^{X,Y}"] \arrow[dd, "f \times g"] & \\
X \arrow[dd, "f"'] & & Y \arrow[dd, "g"] \\
 & X' \times Y' \arrow[dl, "p_1^{X',Y'}"'] \arrow[dr, "p_2^{X',Y'}"] \arrow[dd, "f' \times g'"] & \\
X' \arrow[dd, "f'"'] & & Y' \arrow[dd, "g'"] \\
 & X'' \times Y'' \arrow[dl, "p_1^{X'',Y''}"'] \arrow[dr, "p_2^{X'',Y''}"] & \\
X'' & & Y''
\end{tikzcd}$$

By leaving out the middle part of this diagram, we get the following commutative diagram:

$$\begin{tikzcd}
 & X \times Y \arrow[dl, "p_1^{X,Y}"'] \arrow[dr, "p_2^{X,Y}"] \arrow[dd, "(f' \times g') \circ (f \times g)"] & \\
X \arrow[dd, "f' \circ f"'] & & Y \arrow[dd, "g' \circ g"] \\
 & X'' \times Y'' \arrow[dl, "p_1^{X'',Y''}"'] \arrow[dr, "p_2^{X'',Y''}"] & \\
X'' & & Y''
\end{tikzcd}$$

The commutativity of this diagram shows that the composite $(f' \times g') \circ (f \times g)$ satisfies the defining property of $(f' \circ f) \times (g' \circ g)$, whence

$$(f' \times g') \circ (f \times g) = (f' \circ f) \times (g' \circ g)\,.$$

We have therefore the chain of equalities

$$\begin{aligned} P(f',g')\circ P(f,g) &= (f'\times g')\circ(f\times g)\\ &= (f'\circ f)\times(g'\circ g)\\ &= P(f'\circ f, g'\circ g)\\ &= P((f',g')\circ(f,g)). \end{aligned}$$

We have thus constructed a functor P from $\mathscr{A}\times\mathscr{A}$ to $\mathscr{A}$, given by on objects by $P(X,Y) = X\times Y$ and on morphisms by $P(f,g) = f\times g$.

Exercise 5.3.9

For every object A of $\mathscr{A}$, the map

$$\begin{aligned} \pi_{A,X,Y} : \mathscr{A}(A, X\times Y) &\longrightarrow \mathscr{A}(A,X)\times\mathscr{A}(A,Y),\\ h &\longmapsto (p_1^{X,Y}\circ h,\ p_2^{X,Y}\circ h) \end{aligned}$$

is a bijection by the universal property of the product $X\times Y$. We need to show that this bijection is natural in A, X and Y.

For every morphism

$$f : A' \longrightarrow A$$

in $\mathscr{A}$, the resulting diagram

$$\begin{tikzcd} \mathscr{A}(A, X\times Y) \arrow[r, "\pi_{A,X,Y}"] \arrow[d, "f^*"'] & \mathscr{A}(A,X)\times\mathscr{A}(A,Y) \arrow[d, "f^*\times f^*"] \\ \mathscr{A}(A', X\times Y) \arrow[r, "\pi_{A',X,Y}"] & \mathscr{A}(A',X)\times\mathscr{A}(A',Y) \end{tikzcd}$$

commutes, because

$$\begin{aligned} (f^*\times f^*)(\pi_{A,X,Y}(h)) &= (f^*\times f^*)(p_1^{X,Y}\circ h,\ p_2^{X,Y}\circ h)\\ &= (f^*(p_1^{X,Y}\circ h),\ f^*(p_2^{X,Y}\circ h))\\ &= (p_1^{X,Y}\circ h\circ f,\ p_2^{X,Y}\circ h\circ f)\\ &= \pi_{A',X,Y}(h\circ f)\\ &= \pi_{A',X,Y}(f^*(h)) \end{aligned}$$

for every element h of $\mathscr{A}(A, X \times Y)$. This shows the naturality of the transformation $(\pi_{A,X,Y})_{A,X,Y}$ in A.

The naturality in both X and Y is equivalent to the naturality in (X, Y). Let therefore

$$(f, g) \colon (X, Y) \longrightarrow (X', Y')$$

be a morphism in $\mathscr{A} \times \mathscr{B}$. The resulting diagram

$$\begin{tikzcd}
\mathscr{A}(A, X \times Y) \arrow[r, "\pi_{A,X,Y}"] \arrow[d, "(f\times g)_*"'] & \mathscr{A}(A, X) \times \mathscr{A}(A, Y) \arrow[d, "f_* \times g_*"] \\
\mathscr{A}(A, X' \times Y') \arrow[r, "\pi_{A,X',Y'}"] & \mathscr{A}(A, X') \times \mathscr{A}(A, Y')
\end{tikzcd}$$

commutes, because

$$\begin{aligned}
(f_* \times g_*)(\pi_{A,X,Y}(h)) &= (f_* \times g_*)(p_1^{X,Y} \circ h,\ p_2^{X,Y} \circ h) \\
&= (f \circ p_1^{X,Y} \circ h,\ g \circ p_2^{X,Y} \circ h) \\
&= (p_1^{X',Y'} \circ (f \times g) \circ h,\ p_2^{X',Y'} \circ (f \times g) \circ h) \\
&= \pi_{A,X',Y'}((f \times g) \circ h) \\
&= \pi_{A,X',Y'}((f \times g)_*(h))
\end{aligned}$$

for every element h of $\mathscr{A}(A, X \times Y)$. This shows the naturality of $(\pi_{A,X,Y})_{A,X,Y}$ in (X, Y), and thus in both X and Y.

Exercise 5.3.10

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories and let F be a functor from $\mathscr{A}$. Let $\mathbf{I}$ be a small category and let D be a diagram of shape $\mathbf{I}$ in $\mathscr{A}$. For every cone $C = (A, (p_I)_I)$ of the diagram D, we denote the resulting cone $(F(A), (F(p_I))_I)$ of the diagram $F \circ D$ by $F(C)$.

Suppose now that the functor F creates limits. We need to show that the functor F also reflects limits. For this, let C be a cone on D such that $F(C)$ is a limit cone on $F \circ D$. We need to show that C is a limit cone on D.

There exists by assumption a unique cone L on D with $F(L) = F(C)$, and this cone L is a limit cone on D. We have $L = C$ by the uniqueness of L, whence C is a limit cone on D.

Exercise 5.3.11

(a)

Let $\mathbf{I}$ be a small category and let D be a diagram of shape $\mathbf{I}$ in $\mathbf{Grp}$. For every object J of $\mathbf{I}$ let pr_J be the projection from the product $\prod_{I \in \mathrm{Ob}(\mathbf{I})} U(D(I))$ onto its J-th factor $U(D(J))$.

An explicit description for the limit of $U \circ D$ is given by the set

$$L' := \left\{ (x_I)_I \in \prod_{I \in \mathrm{Ob}(\mathbf{I})} U(I) \,\middle|\, \begin{array}{l} x_K = D(u)(x_J) \text{ for every} \\ \text{morphism } u \colon J \to K \text{ in } \mathbf{I} \end{array} \right\},$$

and for every object I of $\mathbf{I}$ the projections p'_I from L' to $U(D(I))$ is given by restrictions of the projections pr_I. We need to show that there exists a unique group structure on the set L' that makes each projection p'_I into a homomorphism of groups from the resulting group L to the group $D(I)$. We shall then denote this homomorphism of groups by p_I instead of p'_I (so that the map p'_I is the image of p_I under the forgetful functor U). Afterwards, we need to show that $(L, (p_I)_I)$ is a limit cone on the diagram D.

We first want to uniquely endow the set L' with a group structure – resulting in a group L – such that for each object I of $\mathbf{I}$ the projection map p'_I is a homomorphism of groups – then denoted by p_I – from L to $D(I)$. If such a group structure exists, then we must have for every two elements x and y of L with $x = (x_I)_I$ and $(y_I)_I$ the equalities

$$p_I(x \cdot y) = p_I(x) \cdot p_I(y) = x_I \cdot y_I \,.$$

This means that the group structure of L needs to be given by

$$(x_I)_I \cdot (y_I)_I = (x_I \cdot y_I)_I \,. \tag{5.10}$$

This shows the uniqueness of the desired group structure.

To show that (5.10) results in a well-defined group structure on L', we only need to show that L' is a subgroup of $\prod_{I \in \mathrm{Ob}(\mathbf{I})} D(I)$. In other words, we need to show that L' contain the identity element, and that for any two of its elements x and y, their product $x \cdot y$ is again contained in L'.

- The identity element of $\prod_{I \in \mathrm{Ob}(\mathbf{I})} D(I)$ is given by $1 := (1_{D(I)})_I$. This element satisfies for every morphism

$$u \colon J \longrightarrow K$$

of **I** the chain of equalities

$$D(u)(\mathrm{pr}_J(1)) = D(u)(1_{D(J)}) = 1_{D(K)} = \mathrm{pr}_K(1)\,,$$

and is therefore contained in L'.

- For every two elements $x = (x_I)_I$ and $y = (y_I)_I$ on L', we have for every morphism

  $$u\colon\; J \longrightarrow K$$

  of **I** the chain of equalities

  $$\begin{aligned} D(u)(\mathrm{pr}_J(x \cdot y)) &= D(u)(\mathrm{pr}_J(x) \cdot \mathrm{pr}_J(y)) \\ &= D(u)(x_J \cdot y_J) \\ &= D(u)(x_J) \cdot D(u)(y_J) \\ &= x_K \cdot y_K \\ &= \mathrm{pr}_K(x) \cdot \mathrm{pr}_K(y) \\ &= \mathrm{pr}_K(x \cdot y)\,. \end{aligned}$$

  This shows that the product $x \cdot y$ is again contained in L'.

We have now overall shown that there exists a unique group structure on the set L' – making it into a group L – such that for every object I of **I**, the projection map p'_I from L' to $U(D(I))$ is a homomorphism of groups from L to $D(I)$ – which we shall denote by p_I.

Next, we want to show that $(L, (p_I)_I)$ is a cone on D. To this end, we need to show that

$$D(u) \circ p_J = p_K$$

for every morphism

$$u\colon\; J \longrightarrow K$$

in **I**. It suffices to show that

$$U(D(u) \circ p_J) = U(p_K)$$

because the forgetful functor U is faithful. We also recall that

$$U(D(u)) \circ p'_J = p'_K$$

because $(L', (p'_I)_I)$ is a (limit) cone on $U \circ D$. It follows that

$$U(D(u) \circ p_J) = U(D(u)) \circ U(p_J) = U(D(u)) \circ p'_J = p'_K = U(p_K)\,,$$

as desired. We have thus shown that $(L, (p_I)_I)$ is a cone on the diagram D.

It remains to show that the cone $(L, (p_I)_I)$ is already a limit cone. To show this, let $(C, (q_I)_I)$ be another cone on D. We need to show that there exists a unique homomorphism of groups f from C to L with $p_I \circ f = q_I$ for every object I of $\mathbf{I}$.

By applying the forgetful functor U to the cone $(C, (q_I)_I)$ of D, we arrive at the cone $(C', (q'_I)_I)$ of $U \circ D$. (In other words, $C' = U(C)$ and $q'_I = U(q_I)$ for every object I of $\mathbf{I}$.) The cone $(L, (p'_I)_I)$ is, by assumption, a limit cone on $U \circ D$. There hence exists a unique set-theoretic map f' from C' to L' with

$$p'_I \circ f' = q'_I$$

for every object I of $\mathbf{I}$. It suffices to show in the following that f' is already a homomorphism of groups from C to L.

To show this, let x and y be two elements of C. We have for every object I of $\mathbf{I}$ the chain of equalities

$$\begin{aligned} p'_I(f'(x \cdot y)) &= q'_I(x \cdot y) \\ &= q'_I(x) \cdot q'_I(y) \\ &= p'_I(f'(x)) \cdot p'_I(f'(y)) \\ &= p'_I(f'(x) \cdot f'(y))\,, \end{aligned}$$

and therefore altogether the equality

$$f'(x \cdot y) = f'(x) \cdot f'(y)\,.$$

This shows that f' is indeed a homomorphism of groups from C to L.

(b)

The same argumentation goes through for any kind of “category of algebras”.

Exercise 5.3.12

Let D be a diagram of shape $\mathbf{I}$ in $\mathscr{A}$.

The induced diagram $F \circ D$ in $\mathscr{B}$ is again of shape $\mathbf{I}$, and therefore admits a limit cone $(L', (p'_I)_I)$. The functor F creates limits by assumption. There hence exists a unique cone $(L, (p_I)_I)$ of D such that $L' = F(L)$ and $p'_I = F(p_I)$ for every object I of $\mathbf{I}$, and this cone $(L, (p_I)_I)$ is a limit cone on D. This shows that the category $\mathscr{A}$ has limits of shape $\mathbf{I}$.

To show that F preserves limits, we make the following observations.

- For every category $\mathscr{A}$ and every diagram D of shape $\mathbf{I}$ in $\mathscr{A}$, we can form the category $\mathbf{Cone}(D)$ of cones over D.
    - The objects of $\mathbf{Cone}(D)$ are cones over D.
    - A morphism from a cone $(C, (p_I)_I)$ to a cone $(C', (p'_I)_I)$ in $\mathbf{Cone}(D)$ is a morphism f from C to C' in $\mathscr{A}$ such that $p'_I \circ f = p_I$ for every object I of $\mathbf{I}$.
    - The composition of morphisms in $\mathbf{Cone}(D)$ is the composition of morphisms in $\mathscr{A}$.
    - For every cone $(C, (p_I)_I)$ over D, its identity morphism in $\mathbf{Cone}(D)$ is given by the identity morphism of C in $\mathscr{A}$.

  (We have thus a forgetful functor from $\mathbf{Cone}(D)$ to $\mathscr{A}$ that assigns to each cone its vertex.)
- A cone over a diagram D is a limit cone over D if and only if it is terminal in $\mathbf{Cone}(D)$.
- Let $\mathscr{A}$ and $\mathscr{B}$ be two categories, let F be a functor from $\mathscr{A}$ to $\mathscr{B}$, and let D be a diagram of shape $\mathbf{I}$ in $\mathscr{A}$. The functor F induces a functor $\mathbf{Cone}(F)$ from $\mathbf{Cone}(D)$ to $\mathbf{Cone}(F \circ D)$ as follows:
    - For every cone $(C, (p_I)_I)$ over D, its image under the functor $\mathbf{Cone}(F)$ is the cone $(F(C), (F(p_I))_I)$ over $F \circ D$.
    - Let $(C, (p_I)_I)$ and $(C', (p'_I)_I)$ be two cones over D and let f be a morphism from $(C, (p_I)_I)$ to $(C', (p'_I)_I)$. The image of f under the functor $\mathbf{Cone}(F)$ is $F(f)$.

Let now $\tilde{C} := (\tilde{L}, (\tilde{p}_I)_I)$ be a limit cone on the diagram D. We know that the diagram $C := (L, (p_I)_I)$ is also a limit cone on D. It follows that the cones $\tilde{C}$ and C are both terminal in $\mathbf{Cone}(D)$, and therefore isomorphic in $\mathbf{Cone}(D)$. The resulting cones $\mathbf{Cone}(F)(\tilde{C})$ and $\mathbf{Cone}(F)(C)$ are therefore isomorphic in $\mathbf{Cone}(F \circ D)$. The cone $\mathbf{Cone}(F)(C)$ is given by $(L', (p'_I)_I)$, which is a limit cone on $F \circ D$. Therefore, $\mathbf{Cone}(F)(C)$ is terminal in $\mathbf{Cone}(F \circ D)$. It follows that $\mathbf{Cone}(F)(\tilde{C})$ is also terminal in $\mathbf{Cone}(F \circ D)$, because it is isomorphic to $\mathbf{Cone}(F)(C)$. This means that the cone $\mathbf{Cone}(F)(\tilde{C}) = (F(\tilde{L}), (\tilde{p}_I)_I)$ is a limit cone for the diagram $F \circ D$.

This shows that the functor F preserves limits.

Exercise 5.3.13

(a)

Let S be an arbitrary set. We have for every set S the chain of isomorphisms (of functors)

$$\mathscr{B}(F(S), -) \cong \mathbf{Set}(S, G(-)) \cong \mathbf{Set}(S, -) \circ G\,.$$

The functor G preserves epimorphisms by assumption, so it suffices to show that the functor $\mathbf{Set}(S, -)$ preserves epimorphisms. But the axiom of choice asserts that every epimorphism in **Set** is a split epimorphism, and every functor preserves split epimorphisms.[1]

(b)

The epimorphisms in **Ab** are precisely those homomorphisms of groups that are surjective. If P is a projective object of **Ab**, then it thus follows that for every surjective homomorphism of abelian groups

$$f \colon A \longrightarrow B\,,$$

the induced map

$$f_* \colon \mathbf{Ab}(P, A) \longrightarrow \mathbf{Ab}(P, B)$$

is again surjective.

We consider now the group $P = \mathbb{Z}/2$ and the surjective homomorphism of abelian groups

$$f \colon \mathbb{Z} \longrightarrow \mathbb{Z}/2\,, \quad x \longmapsto [x]\,.$$

The induced map

$$f_* \colon \mathbf{Ab}(\mathbb{Z}/2, \mathbb{Z}) \longrightarrow \mathbf{Ab}(\mathbb{Z}/2, \mathbb{Z}/2)$$

[1] The assertion that $\mathbf{Set}(S, -)$ preserves epimorphisms for every set S, i.e., that every object of **Set** is projective, is in fact equivalent to the axiom of choice.

is not surjective because its domain contains only a single element (namely the zero homomorphism) while its codomain contains two elements (the zero homomorphism and the identity homomorphism).

This tells us that the abelian group $\mathbb{Z}/2$ is not projective in **Ab**.

Remark 5.F. It is a well-know in algebra that in a module category R-**Mod**, where R is some ring, the following conditions on an object P are equivalent:

i. P is projective in R-**Mod**.

ii. For every epimorphism of R-modules

$$p \colon A \longrightarrow B$$

and every homomorphism of R-modules

$$f \colon P \longrightarrow B\,,$$

there exists a lift of f along p. More explicitly, there exists a homomorphism of R-modules g from P to A with $f = p \circ g$, i.e., such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "f"] \\ A \arrow[r, "p"] & B \end{tikzcd}$$

iii. Every epimorphism of R-modules with codomain P splits.

iv. Every short exact sequence of R-modules that ends in P splits.

v. P is (isomorphic to) a direct summand of a free R-module.

Our above (counter)example is based on the observation that $\mathbb{Z}/2$ cannot be a direct summand of a free $\mathbb{Z}$-module, since $\mathbb{Z}/2$ is a non-trivial torsion module but free $\mathbb{Z}$-modules have only trivial torsion (because $\mathbb{Z}$ is an integral domain).

(c)

More explicitly, an object I of a category $\mathscr{B}$ is injective if and only if the contravariant functor $\mathscr{B}(-, I)$ turns monomorphisms (in $\mathscr{B}$) into epimorphisms (in **Set**). Even more explicitly: for every monomorphism

$$m\colon\ B \longrightarrow B'$$

in $\mathscr{B}$, every morphism

$$f\colon\ B \longrightarrow I$$

extends to a morphism g from B' to I, in the sense that $g' \circ m = f$, i.e., such that the following diagram commutes:

$$\begin{tikzcd} & I \\ A \arrow[ur, "f"] \arrow[r, "m"'] & A' \arrow[u, dashed, "g"'] \end{tikzcd}$$

Let us show that every object on $\mathbf{Vect}_{\Bbbk}$ is injective. For this, let W be a $\Bbbk$-vector space and let

$$m\colon U \longrightarrow V$$

be a monomorphism of $\Bbbk$-vector spaces. This means that m is an injective and $\Bbbk$-linear map from U to V. Thanks to the axiom of choice, (or rather, Zorn's lemma,) there exists a linear subspace V' of V with

$$V = \operatorname{im}(m) \oplus V' .$$

It follows that there exists a linear map e from V to U with $e \circ m = 1_U$; one such map is given by

$$e(m(u) + v') = u$$

for all elements u and v' of U and V' respectively.

It follows for every linear map f from U to W that the composite $g := f \circ e$ is a linear map from V to W with

$$g \circ m = f \circ e \circ m = f \circ 1_U = f .$$

We have thus proven that W is injective in $\mathbf{Vect}_{\Bbbk}$.

Let us now show that the abelian group $\mathbb{Z}$ is non-injective in **Ab**. To see this, we let i be the inclusion map from $2\mathbb{Z}$ to $\mathbb{Z}$, and consider the homomorphism of groups

$$f : 2\mathbb{Z} \longrightarrow \mathbb{Z}, \quad \mathbb{Z}, \quad n \longmapsto \frac{n}{2}.$$

There exists no homomorphism of groups g from $\mathbb{Z}$ to $\mathbb{Z}$ with $g \circ i = f$, i.e., such that the diagram

$$\begin{tikzcd} & \mathbb{Z} \\ 2\mathbb{Z} \arrow[ur, "f"] \arrow[r, "i"] & 2\mathbb{Z} \arrow[u, dashed, "g"'] \end{tikzcd}$$

commutes. Indeed, if such a homomorphism g were to exist, then the element $x := g(1)$ of $\mathbb{Z}$ would need to satisfy the equations

$$2x = 2g(1) = g(2) = f(i(2)) = f(2) = \frac{2}{2} = 1.$$

But no element x of $\mathbb{Z}$ satisfies the equation $2x = 1$.

We have thus shown that the abelian group $\mathbb{Z}$ is not injective in **Ab**.

Remark 5.G. Let R be a ring.

Baer's criterion asserts that an R-module I is injective if and only if for every ideal J of R, every homomorphism of R-modules from J to I extends to a homomorphism of R-modules from R to I.

If R is a principal ideal domain, then this means that an R-module is injective if and only if it is divisible.

Our above (counter)example is based on the observation that $\mathbb{Z}$ is not divisible as a $\mathbb{Z}$-module, since multiplication by 2 is not surjective.

Chapter 6

Adjoints, representables and limits

6.1 Limits in terms of representables and adjoints

Exercise 6.1.5

The category of diagrams of shape $\mathbf{I}$ in $\mathscr{A}$ – i.e., the functor category $[\mathbf{I}, \mathscr{A}]$ – is isomorphic to the product category $\mathscr{A} \times \mathscr{A}$. For every object A of $\mathscr{A}$, the object of $\mathscr{A} \times \mathscr{A}$ corresponding to the constant functor $\Delta(A)$ is the pair (A, A).

A cone on an object (A, B) of $\mathscr{A} \times \mathscr{A}$ is an object Q of $\mathscr{A}$ together with two morphisms

$$q_1 : Q \longrightarrow A, \quad q_2 : Q \longrightarrow B.$$

A limit cone is thus an object P of $\mathscr{A}$ together with two morphisms

$$p_1 : P \longrightarrow A, \quad p_2 : P \longrightarrow B$$

such that for every other cone (Q, q_1, q_2) as above, there exists a unique morphism f from Q to P with $p_1 \circ f = q_1$ and $p_2 \circ f = q_2$. In other words, (P, p_1, p_2) is precisely a product of the two objects A and B.

Proposition 6.1.4 gives us the functoriality of the product $(-) \times (-)$ from Exercise 5.3.8 It also tells us that this product functor

$$(-) \times (-) : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$$

is right adjoint to the diagonal functor

$$\Delta : \mathscr{A} \longrightarrow \mathscr{A} \times \mathscr{A}.$$

(We had already seen this for $\mathscr{A} = \mathbf{Set}$ as part of Exercise 3.1.1.)

Exercise 6.1.6

Let G be a group. We may identify the functor category $[G, \mathbf{Set}]$ with the category of G-sets, which we shall denote by G-**Set**. We recall that an element x of a G-set is called **invariant** if

$$g \cdot x = x \qquad \text{for every } g \in G\,.$$

A G-set X is trivial if each element of G acts by the identity on X. In other words, every element of X needs to be G-invariant.

For every G-set X we can consider its set of invariants

$$X^G := \{x \in X \mid x \text{ is } G\text{-invariant}\}\,.$$

This is the largest subset of X on which G act trivially. We can dually consider its set of coinvariants, denoted by X_G, which is the largest quotient of X on which G act trivially. It can be constructed as the set

$$X_G := X/\!\sim$$

where the equivalence relation $\sim$ of X is generated by $x \sim g \cdot x$ with $x \in X$ and $g \in G$. The set X_G can equivalently be described as the set of G-orbits of X.

Let X be a G-set. When regarded as a diagram of shape G, a cone on X is a set S together with a map

$$f \colon S \longrightarrow X$$

such that $g \cdot f(s) = f(s)$ for every element s of S. In other words, the image of the map f must be contained in the set of invariants X^G.

It follows that the set X^G together with the inclusion map from X^G to X is a limit cone on X. We find dually that the set X_G together with the quotient map from X to X_G is a colimit cocone on X.

The diagonal functor

$$\Delta \colon \mathbf{Set} \longrightarrow [G, \mathbf{Set}]$$

corresponds to the functor

$$T \colon \mathbf{Set} \longrightarrow G\text{-}\mathbf{Set}$$

that regards any set as a trivial G-set. We get from Proposition 6.1.4 the adjunctions

$$(-)_G \dashv T \dashv (-)^G\,.$$

We have already encountered these adjunctions in (our solution to) Exercise 2.1.16.

6.2 Limits and colimits of presheaves

Exercise 6.2.20

(a)

It follows from Corollary 6.2.6 that the functor category $[\mathbf{A}, \mathscr{S}]$ also has pullbacks. According to Lemma 5.1.32, the natural transformation α is a monomorphism in $[\mathbf{A}, \mathscr{S}]$ if and only if the diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "1_X"'] & X \arrow[d, "\alpha"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd} \tag{6.1}$$

is a pullback diagram in $[\mathbf{A}, \mathscr{S}]$.

It follows from Corollary 6.2.6 that if the above diagram is a pullback diagram, then for every object A of $\mathbf{A}$, the resulting diagram

$$\begin{tikzcd} X(A) \arrow[r, "1_{X(A)}"] \arrow[d, "1_{X(A)}"'] & X(A) \arrow[d, "\alpha_A"] \\ X(A) \arrow[r, "\alpha_A"'] & Y(A) \end{tikzcd} \tag{6.2}$$

is a pullback diagram in $\mathscr{S}$. The converse is also true, by Theorem 6.2.5. Therefore, the diagram (6.1) is a pullback diagram in $[\mathbf{A}, \mathscr{S}]$ if and only if for every object A in $\mathbf{A}$, the diagram (6.2) is a pullback diagram in $\mathscr{S}$.

We hence have by Lemma 5.1.32 the following chain of equivalences:

$$\begin{aligned} & \alpha \text{ is a natural transformation} \\ \iff & \text{the diagram (6.1) is a pullback diagram in } [\mathbf{A}, \mathscr{S}] \\ \iff & \text{the diagram (6.2) is a pullback diagram in } \mathscr{S} \text{ for every object } A \text{ of } \mathbf{A} \\ \iff & \text{the morphism } \alpha_A \text{ is a monomorphism in } \mathscr{S} \text{ for every object } A \text{ of } \mathbf{A}\,. \end{aligned}$$

Therefore, the natural transformation α is a monomorphism in $[\mathbf{A}, \mathscr{S}]$ if and only if each of its components α_A is a monomorphism in $\mathscr{S}$.

We can similarly use the dual of Lemma 5.1.32 to show the following proposition:

> *Let* $\mathbf{A}$ *be a small category and let* $\mathscr{S}$ *be a locally small category with pushouts. The epimorphisms in* $[\mathbf{A}, \mathscr{S}]$ *are precisely those natural transformations that are an epimorphism in each component.*

(b)

The category **Set** admits both pullback and pushouts. The monomorphisms in **Set** are precisely those maps that are injective, and the epimorphisms are precisely those maps that are surjective.

It follows that the monomorphisms in $[\mathbf{A}, \mathbf{Set}]$ are precisely those natural transformations whose every component is injective. Similarly, the epimorphisms in $[\mathbf{A}, \mathbf{Set}]$ are precisely those natural transformations whose every component is surjective.

(c)

Monomorphisms

Suppose first that for every object A of $\mathbf{A}$, the morphism α_A is a monomorphism in $\mathscr{S}$. Let

$$\beta, \beta' : Z \longrightarrow X$$

be two morphisms in $[\mathbf{A}, \mathscr{S}]$ (i.e., natural transformations between the functors Z and X) with $\alpha \circ \beta = \alpha \circ \beta'$. This means that for every object A of $\mathbf{A}$, we have the chain of equalities

$$\alpha_A \circ \beta_A = (\alpha \circ \beta)_A = (\alpha \circ \beta')_A = \alpha_A \circ \beta'_A .$$

It follows for every object A of $\mathbf{A}$ that $\beta_A = \beta'_A$ because the morphism α_A is a monomorphism. This shows that $\beta = \beta'$, which in turn shows that α is a monomorphism in $[\mathbf{A}, \mathscr{S}]$.

Suppose conversely that α is a monomorphism. We restrict ourselves to the case that $\mathscr{S} = \mathbf{Set}$. We know from Yoneda's lemma that for every object A of $\mathbf{A}$, the evaluation functor

$$\mathrm{ev}_A : [\mathbf{A}, \mathbf{Set}] \longrightarrow \mathbf{Set}$$

is representable by H_A. But representable functors always preserve monomorphisms.[1] It hence follows that the isomorphic functor ev_A also preserves

[1]A morphism $g : B' \to B''$ in a category $\mathscr{B}$ is a monomorphism if and only if the induced

monomorphisms. Therefore, the component $\alpha_A = \mathrm{ev}_A(\alpha)$ is a monomorphism in **Set** for every object A of **A**.

Epimorphisms

If each component of α is an epimorphism in $\mathscr{S}$, then we can proceed in the same way as above to see that α is an epimorphism in $[\mathbf{A}, \mathscr{S}]$.

It remains to show that for an epimorphism in $[\mathbf{A}, \mathbf{Set}]$, all of its components are epimorphisms in **Set**. However, the author doesn't know how to do this without using part (a) of this exercise.

Remark 6.A. For more information on the missing part of the above solution, we refer to [MSE22].

Exercise 6.2.21

(a)

Let α be an isomorphism from $X + Y$ to H_A. Let

$$i : X \Longrightarrow X + Y , \quad j : Y \Longrightarrow X + Y$$

be natural transformations that realize $X + Y$ as a coproduct of X and Y, and let

$$\beta \coloneqq \alpha \circ i : X \Longrightarrow \mathrm{H}_A , \quad \gamma \coloneqq \alpha \circ j : Y \Longrightarrow \mathrm{H}_A .$$

We know that colimits in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ are computed pointwise. This tells us that for every object A' of $\mathscr{A}$, the two maps

$$i_{A'} : X(A') \longrightarrow (X + Y)(A') , \quad j_{A'} : Y(A') \longrightarrow (X + Y)(A')$$

make the set $(X + Y)(A')$ into a coproduct of the two sets $X(A')$ and $Y(A')$. We know that the coproduct of two sets is given by their disjoint union. The set $(X + Y)(A')$ is therefore the disjoint union of the images of $i_{A'}$ and $j_{A'}$ (and both $i_{A'}$ and $j_{A'}$ are injective). The map $\alpha_{A'}$ is a bijection from $(X + Y)(A')$

map $g_* : \mathscr{B}(B, B') \to \mathscr{B}(B, B'')$ is injective for every object B of $\mathscr{B}$. (This is a direct reformulation of the definition of a monomorphism.) The class of monomorphisms is therefore chosen in precisely such a way that each functor $\mathrm{H}_B = \mathscr{B}(B, -)$ preserves monomorphisms.

to $\mathrm{H}_A(A')$, so it further follows that the set $\mathrm{H}_A(A')$ is the disjoint union of the images of $\alpha_{A'} \circ i_{A'} = \beta_{A'}$ and $\alpha_{A'} \circ j_{A'} = \gamma_{A'}$.

This entails that the set $\mathscr{A}(A, A) = \mathrm{H}_A(A)$ is the disjoint union of the images of $\beta_{A'}$ and $\gamma_{A'}$. The element 1_A of $\mathscr{A}(A, A)$ is therefore contained in precisely on of these two images. We may assume that it is contained in the image of β_A.

We check in the following that the set $Y(A')$ is empty for every object A' of $\mathscr{A}$. We do so by showing the image of $Y(A')$ in $\mathrm{H}_A(A')$ under $\gamma_{A'}$ is empty.

Let x be a preimage of 1_A in $X(A)$ under β_A. (This preimage is in fact unique, since $\beta_A = \alpha_A \circ i_A$ is a composite of two injective maps.) Let A' be an arbitrary object of $\mathscr{A}$ and let f be an element of $\mathrm{H}_A(A')$. This means that f is a morphism from A' to A. Then

$$f = 1_A \circ f = f^*(1_A) = \mathrm{H}_A(f)(1_A) = \mathrm{H}_A(f)(\beta_A(x)) = \beta_{A'}(X(f)(x))$$

by the naturality of β. This shows that all of $\mathrm{H}_A(A')$ is contained in the image of $\beta_{A'}$. But $\mathrm{H}_A(A')$ is the disjoint union of the images of $\beta_{A'}$ and $\gamma_{A'}$. We thus find that the image of $\gamma_{A'}$ is empty, and therefore that $Y(A')$ is empty.

(b)

Let X and Y be two representable presheaves on $\mathscr{A}$. This means that there exists objects A and A' of $\mathscr{A}$ with

$$X \cong \mathrm{H}_A\,, \quad Y \cong \mathrm{H}_{A'}\,.$$

This entails that the sets

$$X(A) \cong \mathrm{H}_A(A) = \mathscr{A}(A, A)\,, \quad Y(A') \cong \mathrm{H}_{A'}(A') = \mathscr{A}(A', A')$$

are non-empty because they contain the elements 1_A and $1_{A'}$ respectively. It follows from part (a) that the functor $X + Y$ cannot be representable.

Exercise 6.2.22

Warning 6.B. For a covariant functor

$$X : \mathscr{A} \longrightarrow \mathbf{Set}\,,$$

its category of elements, denoted by $\mathrm{E}(X)$, is typically defined as follows.

- The objects of $\mathbf{E}(X)$ are pairs (A, x) consisting of an object A of $\mathscr{A}$ and an element x of $X(A)$.
- A morphism in $\mathbf{E}(X)$ from an object (A, x) to an object (A', x') is a morphism f from A to A' in $\mathscr{A}$ with $X(f)(x) = x'$.

A presheaf X on a category $\mathscr{A}$ is a contravariant functor from $\mathscr{A}$ to **Set**, and thus a covariant functor from $\mathscr{A}^{\mathrm{op}}$ to **Set**. The above definition of "category of elements" can therefore be applied to X. This results in the following definition of $\mathbf{E}(X)$.

- The objects of $\mathbf{E}(X)$ are pairs (A, x) consisting of an object A of $\mathscr{A}$ and an element x of $X(A)$.
- A morphism in $\mathbf{E}(X)$ from an object (A, x) to an object (A', x') is a morphism f from A' to A in $\mathscr{A}$ with $X(f)(x) = x'$.

This definition of $\mathbf{E}(X)$ does *not* agree with Definition 6.2.16: the two definitions result in opposite categories.[2]

In the following, we will use Definition 6.2.16, as intended in the book.

In the following, let us refer to the objects of the category $\mathbf{E}(X)$ as the elements of X. By Yoneda's lemma, we have for every object A of $\mathscr{A}$ the bijection

$$[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](\mathrm{H}_A, X) \longrightarrow X(A), \quad h \longmapsto h_A(1_A).$$

We get from this bijection an induced one-to-one correspondence between elements of X and pairs (A, α) consisting of an object A of $\mathscr{A}$ and a natural transformation α from H_A to X. This correspondence is given by the mapping

$$(A, \alpha) \longmapsto (A, \alpha_A(1_A)).$$

Let (A, x) and (A', x') be two elements of X and let (A, α) and (A', α') be the corresponding pairs as above. Every natural transformation from $\mathrm{H}_{A'}$ to $\mathrm{H}_{A'}$ is of the form H_f for a unique morphism f from A to A' because the Yoneda embedding is fully faithful. The resulting diagram

$$\begin{tikzcd} \mathrm{H}_A \arrow[rr, Rightarrow, "\mathrm{H}_f"] \arrow[dr, Rightarrow, "\alpha"'] & & \mathrm{H}_{A'} \arrow[dl, Rightarrow, "\alpha'"] \\ & X & \end{tikzcd}$$

[2]The author is extremely annoyed by this inconsistency.

commutes if and only if

$$\alpha = \alpha' \circ \mathrm{H}_f \,.$$

By Yoneda's lemma, this equality of natural transformations is equivalent to the equality of elements

$$\alpha_A(1_A) = (\alpha' \circ \mathrm{H}_f)_A(1_A)$$

The left-hand side of this equation is the element x, and the right-hand side can be rewritten as

$$\begin{aligned}
(\alpha' \circ \mathrm{H}_f)_A(1_A) &= (\alpha'_A \circ (\mathrm{H}_f)_A)(1_A) \\
&= \alpha'_A((\mathrm{H}_f)_A(1_A)) \\
&= \alpha'_A(\mathrm{H}_{A'}(f)(1_{A'})) && (6.3)\\
&= X(f)(\alpha_{A'}(1_{A'})) && (6.4)\\
&= X(f)(x')\,.
\end{aligned}$$

We use for (6.3) the chain of equalities

$$(\mathrm{H}_f)_A(1_A) = f_*(1_A) = f \circ 1_A = f = 1_{A'} \circ f = f^*(1_{A'}) = \mathrm{H}_{A'}(f)(1_{A'})\,,$$

and (6.4) holds because α is a natural transformation from $\mathrm{H}_{A'}$ to X. The commutativity of the above diagram is therefore equivalent to the equality

$$x = X(f)(x')\,.$$

This equality expresses precisely that f is a morphism from (A, x) to (A', x') in $\mathbf{E}(X)$.

We have altogether an isomorphism

$$Y \Rightarrow X$$

where Y is the Yoneda embedding from $\mathscr{A}$ to $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$, and where we use (the dual of) the notation from Example 2.3.4.[3]

[3]More explicitly, we consider the comma category of the following situation:

$$\begin{tikzcd}
 & \mathbf{1} \arrow[d, "X"] \\
\mathscr{A} \arrow[r, "Y"'] & {[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]}
\end{tikzcd}$$

Exercise 6.2.23

The presheaf X is representable if and only if it admits a universal element. Such a universal element is – by definition – a pair (A, x) consisting of an object A of $\mathscr{A}$ and an element x of $X(A)$ such that for every other object A' of $\mathscr{A}$ and every element x' of $X(A')$, there exists a unique morphism f from A' to A in $\mathscr{A}$ with $X(f)(x) = x'$. But this means precisely that the pair (A, x) is a terminal object in $\mathbf{E}(X)$.

Exercise 6.2.24

Remark 6.C. This exercise is a massive waste of time and the following solution is not proofread in the slightest.

We build up to the case of an arbitrary small category $\mathbf{A}$ by considering first some special cases.

The category A is the one-object category 1

We consider first the special case that $\mathbf{A}$ is the one-object category $\mathbf{1}$, whose unique object we denote by $*$. The category $[\mathbf{1}^{\mathrm{op}}, \mathbf{Set}]$ is isomorphic to $\mathbf{Set}$ via the evaluation functor at $*$. A presheaf X on $\mathbf{1}$ corresponds under this isomorphism to the set $S \coloneqq X(*)$. This isomorphism between $[\mathbf{1}^{\mathrm{op}}, \mathbf{Set}]$ and $\mathbf{Set}$ induces an isomorphism

$$[\mathbf{1}^{\mathrm{op}}, \mathbf{Set}]/X \cong \mathbf{Set}/S\,.$$

We want to describe the category $\mathbf{Set}/S$ as a category of presheaves a suitable small category.

We consider first the case that $S = \{0, 1\}$. An object of $\mathbf{Set}/S$ is a pair (A, σ) consisting of a set A and a function σ from A to S. We know that maps from A to S can be identified with subsets of A, with the function σ corresponding to the preimage $\sigma^{-1}(1)$. A morphism from (A, σ) to (A', σ') in $\mathbf{Set}/S$ is a map f from A to A' subject to the commutativity of the following diagram:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\sigma"'] & & A' \arrow[dl, "\sigma'"] \\ & S & \end{tikzcd}$$

The commutativity of this diagram can equivalently be expressed via the sets $\sigma^{-1}(1)$ and $(\sigma')^{-1}(1)$ as

$$f^{-1}((\sigma')^{-1}(1)) = \sigma^{-1}(1)\,.$$

We hence find that the category $\mathbf{Set}/S$ is isomorphic to the following auxiliary category $\mathscr{S}$: Objects of $\mathscr{S}$ are pairs (A, A_1) consisting of a set A and a subset A_1 of A. A morphism from (A, A_1) to (A', A'_1) in $\mathscr{S}$ is a map f from A to A' with $f^{-1}(A'_1) = A_1$.

We can generalize this description of $\mathbf{Set}/S$ to the case that S is an arbitrary set. We find that $\mathbf{Set}/S$ is isomorphic to the following auxiliary category $\mathscr{P}_S$:

- Objects of $\mathscr{P}_S$ are pairs $(A, (A_s)_s)$ consisting of a set A and a family $(A_s)_s$ of subsets A_s of A, indexed over the elements s of S, such that A is the disjoint union of the sets A_s.
- A morphism from $(A, (A_s)_s)$ to $(A', (A'_s)_s)$ in $\mathscr{P}_S$ is a map f from A to A' such that
$$A_s = f^{-1}(A'_s) \qquad \text{for every } s \in S\,.$$
The set A is the disjoint union of the sets A_s but at the same time also the disjoint union of the sets $f^{-1}(A'_s)$. Therefore,
$$\begin{aligned} & A_s = f^{-1}(A'_s) \text{ for every } s \in S \\ \iff\ & A_s \subseteq f^{-1}(A'_s) \text{ for every } s \in S \\ \iff\ & f(A_s) \subseteq A'_s \text{ for every } s \in S\,. \end{aligned}$$
This means that the morphisms from $(A, (A_s)_s)$ to $(A', (A'_s)_s)$ in $\mathscr{P}_S$ are precisely those maps from A to A' that restrict for every index s to a map from A_s to A'_s.

We have a functor
$$\tilde{F} : \mathscr{P}_S \longrightarrow \mathbf{Set}^S$$
that splits apart an object $(A, (A_s)_s)$ into its subsets A_s. More explicitly, this functor is given on every object $(A, (A_s)_s)$ of $\mathscr{P}_S$ by

$$\tilde{F}\Big((A, (A_s)_s)\Big) = (A_s)_s\,,$$

and on every morphism f from $(A, (A_s)_s)$ to $(A', (A'_s)_s)$ by

$$\tilde{F}(f) = (f_s)_s \qquad \text{with} \qquad f_s = f\big|_{A_s}^{B_s}\,.$$

We have also a functor

$$\tilde{G} : \mathbf{Set}^S \longrightarrow \mathscr{P}_S$$

given by the disjoint union $\coprod_{s \in S}(-)$. More explicitly, this functor is given on every object $(A_s)_s$ of $\mathbf{Set}^S$ by

$$\tilde{G}\Big((A_s)_s\Big) = \left(\coprod_{s \in S} A_s, (\tilde{A}_s)_s \right)$$

where $\tilde{A}_t$ is the image of A_t in $\coprod_{s \in S} A_s$, and it is given on morphisms by

$$\tilde{G}\Big((f_s)_s\Big) = \coprod_{s \in S} f_s \,.$$

These two functors $\tilde{F}$ and $\tilde{G}$ satisfy

$$\tilde{G} \circ \tilde{F} \cong 1_{\mathscr{P}_S} \,, \quad \tilde{F} \circ \tilde{G} \cong 1_{\mathbf{Set}^S} \,,$$

and thus form an equivalence between the two categories $\mathscr{P}_S$ and $\mathbf{Set}^S$.

Together with the isomorphism between $\mathbf{Set}/S$ and $\mathscr{P}_S$ we arrive at an equivalence of categories

$$\mathbf{Set}/S \simeq \mathbf{Set}^S \,.$$

We can furthermore identify $\mathbf{Set}^S$ with the functor category $[S^{\mathrm{op}}, \mathbf{Set}]$ when considering the set S as a discrete category. In this way, we arrive overall at the chain of equivalences

$$[\mathbf{1}^{\mathrm{op}}, \mathbf{Set}]/X \cong \mathbf{Set}/S \cong \mathscr{P}_S \simeq \mathbf{Set}^S \cong [S^{\mathrm{op}}, \mathbf{Set}] \,,$$

where the set S corresponds to the presheaf X via $S = X(*)$.

Let F be the composite of the above equivalences. We can work out an explicit description of F as follows.

1. An object of $[\mathbf{1}^{\mathrm{op}}, \mathbf{Set}]/X$ is a pair (Y, α) consisting of a presheaf Y on $\mathbf{1}$ and a natural transformation α from Y to X.
2. Under the isomorphism $[\mathbf{1}^{\mathrm{op}}, \mathbf{Set}]/X \cong \mathbf{Set}/S$, the pair (Y, α) now corresponds to the pair $(Y(*), \alpha_*)$, consisting of the set $Y(*)$ and the map α_* from $Y(*)$ to $X(*) = S$.
3. Under the equivalence $\mathbf{Set}/S \simeq \mathscr{P}_S$, this pair $(Y(*), \alpha_*)$ further corresponds to the pair $(Y(*), (\alpha_*^{-1}(s))_s)$.

4. Under the equivalence $\mathscr{P}_S \simeq \mathbf{Set}^S$ we arrive at the object $(\alpha_*^{-1}(s))_s$.
5. Under the isomorphism $\mathbf{Set}^S \cong [S^{\mathrm{op}}, \mathbf{Set}]$ we arrive at the presheaf $F(Y,\alpha)$ on S (where we view S as a discrete category).[4] It is given by

$$F(Y,\alpha)(s) = \alpha_*^{-1}(s)$$

for every element s of S. (We don't need to worry about the action of $F(Y,\alpha)$ on morphisms of S because the category S is discrete.)

We can similarly figure out how F acts on morphisms. A morphism in the category $[\mathbf{1}^{\mathrm{op}}, \mathbf{Set}]$ from (Y,α) to (Y',α') is a natural transformation β from Y to Y' with $\alpha' \circ \beta = \alpha$, i.e., such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[rr, Rightarrow, "\beta"] \arrow[dr, Rightarrow, "\alpha"'] & & Y' \arrow[dl, Rightarrow, "\alpha'"] \\ & X & \end{tikzcd}$$

The resulting natural transformation $F(\beta)$ from $F(Y,\alpha)$ to $F(Y',\alpha')$ is given by

$$F(\beta) = (F(\beta)_s)_s$$

where for every index s, the component $F(\beta)_s$ is the restriction of the map β_* to a map from $F(Y,\alpha)(s) = \alpha_*^{-1}(s)$ to $F(Y',\alpha')(s) = (\alpha'_*)^{-1}(s)$.

The category A is discrete

We consider now instead of an arbitrary small category $\mathbf{A}$ a small category $\mathbf{D}$ that is discrete. We may think about the category $\mathbf{D}$ as the disjoint union of copies of $\mathbf{1}$, with one copy for each object of $\mathscr{D}$. A presheaf X on $\mathbf{D}$ is then a sum of presheaves X_d on $\mathbf{1}$, given by $X_d(*) = X(d)$ for each object d of $\mathbf{D}$. We can see through the power of abstract nonsense that

$$\begin{aligned} [\mathbf{D}^{\mathrm{op}}, \mathbf{Set}]/X &= [\mathbf{D}, \mathbf{Set}]/X \\ &\cong \left[\coprod_{d \in \mathrm{Ob}(\mathbf{D})} \mathbf{1},\ \mathbf{Set} \right] / X \end{aligned}$$

[4] We image of (Y,α) under F ought to be denoted by $F((Y,\alpha))$, but we will use the abbreviated notation $F(Y,\alpha)$ for better readability.

$$
\begin{aligned}
&\cong \left(\prod_{d \in \mathrm{Ob}(\mathbf{D})} [\mathbf{1}, \mathbf{Set}] \right) / (X_d)_d \\
&\cong \prod_{d \in \mathrm{Ob}(\mathbf{D})} \Big([\mathbf{1}, \mathbf{Set}] / X_d \Big) \\
&= \prod_{d \in \mathrm{Ob}(\mathbf{D})} \Big([\mathbf{1}^{\mathrm{op}}, \mathbf{Set}] / X_d \Big) \\
&\simeq \prod_{d \in \mathrm{Ob}(\mathbf{D})} [(\mathbf{B}_d)^{\mathrm{op}}, \mathbf{Set}] \\
&\simeq \left[\coprod_{d \in \mathrm{Ob}(\mathbf{D})} (\mathbf{B}_d)^{\mathrm{op}},\ \mathbf{Set} \right] \\
&\cong \left[\left(\coprod_{d \in \mathrm{Ob}(\mathbf{D})} \mathbf{B}_d \right)^{\mathrm{op}},\ \mathbf{Set} \right]
\end{aligned}
$$

for some suitable small categories $\mathbf{B}_d$. We have seen above that the categories $\mathbf{B}_d$ can be chosen as $\mathbf{B}_d = X(d)$, viewed as discrete categories. The desired small category $\mathbf{B}$ with

$$[\mathbf{D}^{\mathrm{op}}, \mathbf{Set}]/X \simeq [\mathbf{B}^{\mathrm{op}}, \mathbf{Set}]$$

should therefore be choosable as the set $\coprod_{d \in \mathrm{Ob}(\mathbf{D})} X(d)$ viewed as a discrete category.

Let us make this more explicit. We consider this set

$$B \coloneqq \coprod_{d \in \mathrm{Ob}(\mathbf{D})} X(d)$$

as a discrete category. Let (Y, α) be an object of the category $[\mathbf{D}^{\mathrm{op}}, \mathbf{Set}]/X$. This means that Y is a presheaf on $\mathbf{D}$ and that α is a natural transformation from Y to X:

$$\alpha : Y \Longrightarrow X .$$

We can define a presheaf $F(Y, \alpha)$ on B via

$$F(Y, \alpha)(d, x) = \alpha_d^{-1}(x)$$

for every element (d, x) of B.[5]

A morphism from (Y, α) to (Y', α') in $[\mathbf{D}^{\mathrm{op}}, \mathbf{Set}]/X$ is a natural transformation β from Y to Y' with $\alpha' \circ \beta = \alpha$, i.e., such that the diagram

$$\begin{tikzcd} Y \arrow[rr, Rightarrow, "\beta"] \arrow[dr, Rightarrow, "\alpha"'] & & Y' \arrow[dl, Rightarrow, "\alpha'"] \\ & X & \end{tikzcd}$$

commutes. This means that we have the following commutative diagram for every object d of D:

$$\begin{tikzcd} Y(d) \arrow[rr, "\beta_d"] \arrow[dr, "\alpha_d"'] & & Y'(d) \arrow[dl, "\alpha'_d"] \\ & X(d) & \end{tikzcd}$$

The map β_d therefore restrict for every element x of $X(d)$ to a map between fibres

$$F(\beta)_{(d,x)} : \alpha_d^{-1}(x) \longrightarrow (\alpha'_d)^{-1}(x) .$$

These maps, with the index (d, x) ranging through the set B, define a natural transformation $F(\beta)$ from $F(Y, \alpha)$ to $F(Y', \alpha')$. (We don't need to worry about the naturality of $F(\beta)$ because the category B is discrete.)

Suppose on the other hand that we are given a presheaf Z of B. We then construct a corresponding object $G(B)$ of $[\mathbf{D}^{\mathrm{op}}, \mathbf{Set}]/X$. This to-be-constructed object $G(Z)$ needs to be a pair $(G_0(Z), G_1(Z))$ consisting of a presheaf $G_0(Z)$ on $\mathbf{D}$ and a natural transformation $G_1(Z)$ from $G_0(Z)$ to X.

The presheaf G_0 is constructed as

$$G_0(Z)(d) := \coprod_{x \in X(d)} Z(d, x)$$

for every object d of $\mathbf{D}$. We don't need to worry about the action of $G_0(Z)$ on morphisms because the category $\mathbf{D}$ is discrete.

[5] Recall that for a family of sets $(S_i)_{i \in I}$, the elements of $\coprod_{i \in I} S_i$ can be denoted as pairs (i, s) where i ranges through the index set I and s ranges through the associated set S_i.

We will define the natural transformation $G_1(Z)$ via its components $G_1(Z)_d$, where the index d ranges through the objects of $\mathbf{D}$. The component $G_1(Z)_d$ is the map from $G_0(Z)(d) = \coprod_{x \in X(d)} Z(d, x)$ to $X(d)$ which maps all of $Z(d, x)$ onto the value x of $X(d)$.

These two functors F and G satisfy the isomorphisms

$$G \circ F \cong 1_{[\mathbf{D}^{\mathrm{op}}, \mathbf{Set}]/X}\,, \qquad F \circ G \cong 1_{[B^{\mathrm{op}}, \mathbf{Set}]}\,.$$

The functors F and G hence give an equivalence of categories between the slice category $[\mathbf{D}^{\mathrm{op}}, \mathbf{Set}]/X$ and the functor category $[B^{\mathrm{op}}, \mathbf{Set}]$.

The general case: start

Let now $\mathbf{A}$ be a small category and let X be a presheaf on X. To find the desired category $\mathbf{B}$, we want a small category whose set of objects is given by $\coprod_{A \in \mathrm{Ob}(\mathbf{A})} X(A)$, but who also keeps track of the morphisms in $\mathbf{A}$. We consider for this the category of elements of X, i.e., the category $\mathbf{E}(X)$.

We define first a functor F from $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]/X$ to $[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]$, then a functor G from $[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]$ to $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]/X$, and then we show that these two functors are mutually inverse up to isomorphism.

The general case: construction of F

To construct the functor F on objects, let (Y, α) be an object of $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]/X$. This means that Y is a presheaf Y on $\mathbf{A}$ and that α is a natural transformation α from Y to X. The desired object $F(Y, \alpha)$ needs to be a presheaf on $\mathbf{E}(X)$.

- An object of $\mathbf{E}(X)$ is a pair (A, x) consisting of an object A of $\mathbf{A}$ and an element x of the set $X(A)$. We define the set $F(Y, \alpha)(A, x)$ as

$$F(Y, \alpha)(A, x) := \alpha_A^{-1}(x)\,.$$

- A morphism in $\mathbf{E}(X)$ from an object (A, x) to an object (A', x') is a morphism f from A to A' in $\mathbf{A}$ with $X(f)(x') = x$. We know from the naturality of α that the resulting diagram

$$\begin{tikzcd} Y(A') \arrow[r, "Y(f)"] \arrow[d, "\alpha_{A'}"'] & Y(A) \arrow[d, "\alpha_A"] \\ X(A') \arrow[r, "X(f)"] & X(A) \end{tikzcd}$$

commutes. It follows from the commutativity of this diagram and the equality $X(f)(x') = x$ that the map $Y(f)$ restricts to a map from the preimage $\alpha_{A'}^{-1}(x')$ to the preimage $\alpha_A^{-1}(x)$. We denote this restriction by $F(Y,\alpha)(f)$, so that

$$F(Y,\alpha)(f) : F(Y,\alpha)(A',x') \longrightarrow F(Y,\alpha)(A,x) .$$

We have now constructed the action of $F(Y,\alpha)$ on objects of $\mathbf{E}(X)$ and on morphisms of $\mathbf{E}(X)$. We now check that $F(Y,\alpha)$ is a contravariant functor from $\mathbf{E}(X)$ to $\mathbf{Set}$.

- Let (A,x) be an object on $\mathbf{E}(X)$. The identity morphism of (A,x) is the identity morphism of A, whence $F(Y,\alpha)(1_{(A,x)})$ is the restriction of $Y(1_A)$ to a map from $\alpha_A^{-1}(x)$ to $\alpha_A^{-1}(x)$. But $Y(1_A)$ is $1_{Y(A)}$, whence this restriction is the identity map of $\alpha_A^{-1}(x)$, and thus the identity map of $F(Y,\alpha)(A,x)$. This shows that
$$F(Y,\alpha)(1_{A,x}) = 1_{F(Y,\alpha)(A,x)} .$$
- Let
$$f : (A,x) \longrightarrow (A',x') , \quad g : (A',x') \longrightarrow (A'',x'')$$
be two composable morphisms in $\mathbf{E}(X)$. The map $F(Y,\alpha)(f)$ is the restriction of $Y(f)$ to a map from $\alpha_{A'}^{-1}(x')$ to $\alpha_A^{-1}(x)$, and the map $F(Y,\alpha)(g)$ is similarly the restriction of $Y(g)$ to a map from $\alpha_{A''}^{-1}(x'')$ to $\alpha_{A'}^{-1}(x')$. The composite
$$F(Y,\alpha)(f) \circ F(Y,\alpha)(g)$$
is therefore the restriction of $Y(f) \circ Y(g)$ to a map from $\alpha_{A''}^{-1}(x'')$ to $\alpha_A^{-1}(x)$. But
$$Y(f) \circ Y(g) = Y(g \circ f)$$
by the contravariant functoriality of Y, and the restriction of $Y(g \circ f)$ to a map from $\alpha_{A''}^{-1}(x'')$ to $\alpha_A^{-1}(x)$ is precisely $F(Y,\alpha)(g \circ f)$. We have thus shown that
$$F(Y,\alpha)(f) \circ F(Y,\alpha)(g) = F(Y,\alpha)(g \circ f) .$$

This shows that $F(Y,\alpha)$ is indeed a contravariant functor from $\mathbf{E}(A)$ to $\mathbf{Set}$.

We have thus constructed the desired functor F on the level of objects.

We proceed by constructing the action of F on morphisms. We consider for this a morphism

$$\beta : (Y,\alpha) \longrightarrow (Y',\alpha')$$

in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]/X$. This means that β is a natural transformation from Y to Y' that makes the diagram

$$\begin{tikzcd} Y \arrow[rr, Rightarrow, "\beta"] \arrow[dr, Rightarrow, "\alpha"'] & & Y' \arrow[dl, Rightarrow, "\alpha'"] \\ & X & \end{tikzcd}$$

commute. For every object (A, x) of $\mathbf{E}(X)$ we get the following commutative diagram:

$$\begin{tikzcd} Y(A) \arrow[rr, "\beta_A"] \arrow[dr, "\alpha_A"'] & & Y'(A) \arrow[dl, "\alpha'_A"] \\ & X(A) & \end{tikzcd}$$

It follows from the commutativity of this diagram that the map β_A restricts to a map from $\alpha_A^{-1}(x)$ to $\alpha_{A'}^{-1}(x')$. We denote this restriction by $F(\beta)_{(A,\alpha)}$.

We note that $F(\beta)$ is a natural transformation from $F(Y, \alpha)$ to $F(Y', \alpha')$. Indeed, let

$$f : (A, x) \longrightarrow (A', x')$$

be a morphism in $\mathbf{E}(X)$. This means that f is a morphism from A to A' in $\mathscr{A}$ with $X(f)(x') = x$. It follows from the naturality of β that we have the following commutative diagram:

$$\begin{tikzcd} Y(A) \arrow[r, "\beta_A"] & Y'(A) \\ Y(A') \arrow[u, "Y(f)"] \arrow[r, "\beta_{A'}"] & Y'(A') \arrow[u, "Y'(f)"'] \end{tikzcd}$$

This commutative diagram restricts to the following commutative diagram:

$$\begin{tikzcd} F(Y,\alpha)(A,x) \arrow[r, equal] & \alpha_A^{-1}(x) \arrow[r, "F(\beta)_{(A,x)}"] & (\alpha')_A^{-1}(x) \arrow[r, equal] & F(Y',\alpha')(A,x) \\ F(Y,\alpha)(A',x') \arrow[r, equal] & \alpha_{A'}^{-1}(x') \arrow[u, "F(Y,\alpha)(f)"] \arrow[r, "F(\beta)_{(A',x')}"] & (\alpha')_{A'}^{-1}(x') \arrow[u, "F(Y',\alpha')(f)"'] \arrow[r, equal] & F(Y',\alpha')(A',x') \end{tikzcd}$$

The commutativity of this diagram shows that $F(\beta)$ is indeed a natural transformation from $F(Y,\alpha)$ to $F(Y',\alpha')$.

We have now constructed the action of F on morphisms. We need to check that F is functorial.

- Let (Y,α) be an object of $[\mathbf{A}^{\mathrm{op}},\mathbf{Set}]/X$. The identity morphism of this object is the identity natural transformation of Y, i.e., 1_Y. It follows for every object (A,x) of $\mathbf{E}(X)$ that the map $(1_Y)_A$ is given by $1_{Y(A)}$. The restriction of this map to a map from $F(Y,\alpha)(A,x)$ to $F(Y,\alpha)(A,x)$ is therefore $1_{F(Y,\alpha)(A,x)}$. This shows that

$$F(1_{(Y,\alpha)})_{(A,x)} = 1_{F(Y,\alpha)(A,x)} = (1_{F(Y,\alpha)})_{(A,x)}$$

for every object (A,x) of $\mathbf{E}(X)$, and therefore $F(1_{(Y,\alpha)}) = 1_{F(Y,\alpha)}$.

- Let

$$\beta\colon\ (Y,\alpha) \longrightarrow (Y',\alpha')\,,\quad \beta'\colon\ (Y',\alpha') \longrightarrow (Y'',\alpha'')$$

be two composable morphisms in $[\mathbf{A}^{\mathrm{op}},\mathbf{Set}]/X$. Let (A,x) be an object of $\mathbf{E}(X)$. The map

$$F(\beta)_{(A,x)}\colon\ F(Y,\alpha)(A,x) \longrightarrow F(Y',\alpha')(A,x)$$

is a restriction of β_A, and the map

$$F(\beta')_{(A,x)}\colon\ F(Y',\alpha')(A,x) \longrightarrow F(Y'',\alpha'')(A,x)$$

is similarly a restriction of β'_A. The composite

$$F(\beta')_{(A,x)} \circ F(\beta)_{(A,x)}\colon\ F(Y,\alpha)(A,x) \longrightarrow F(Y'',\alpha'')(A,x)$$

is therefore the restriction of $\beta'_A \circ \beta_A$. But we have $\beta'_A \circ \beta_A = (\beta' \circ \beta)_A$, and the restriction of $(\beta' \circ \beta)_A$ to a map from $F(Y,\alpha)(A,x)$ to $F(Y'',\alpha'')(A,x)$ is precisely $F(\beta' \circ \beta)_{(A,x)}$. We have therefore found that

$$(F(\beta') \circ F(\beta))_{(A,x)} = F(\beta')_{(A,x)} \circ F(\beta)_{(A,x)} = F(\beta' \circ \beta)_{(A,x)}$$

for every object (A,x) of $\mathbf{E}(X)$, and thus altogether

$$F(\beta') \circ F(\beta) = F(\beta' \circ \beta)\,.$$

This shows the functoriality of G.

The general case: construction of G

Let Z be an object of the presheaf category $[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]$. We start by constructing an object $(G_0(Z), G_1(Z))$ of the slice category $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]/X$. This objects needs to consist of an element $G_0(Z)$ of $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ and a natural transformation $G_1(Z)$ from $G_0(Z)$ to X.

We start by constructing $G_0(Z)$, which needs to be a contravariant functor from $\mathbf{A}$ to $\mathbf{Set}$.

- Let A be an object of $\mathbf{A}$. We define the set $G_0(Z)(A)$ as

  $$G_0(Z)(A) := \coprod_{x \in X(A)} Z(A, x),$$

  and denote for every element x of $X(A)$ by

  $$j_{A,x} : Z(A, x) \longrightarrow G_0(Z)(A)$$

  the canonical inclusion map into the x-th summand.
- Let

  $$f : A \longrightarrow A'$$

  be a morphism in $\mathbf{A}$. For every element x' of $X(A')$, this morphism f is then also a morphism

  $$f^{[x']} : (A, X(f)(x')) \longrightarrow (A', x')$$

  in $\mathbf{E}(X)$, and induces therefore a map

  $$Z(f^{[x']}) : Z(A', x') \longrightarrow Z(A, X(f)(x')).$$

  We have hence for every element x' of $X(A')$ the map

  $$j_{A,X(f)(x')} \circ Z(f^{[x']}) : Z(A', x') \longrightarrow G_0(Z)(A).$$

  These maps can now be bundled together into a map

  $$G_0(Z)(f) : G_0(Z)(A') \longrightarrow G_0(Z)(A)$$

  such that

  $$G_0(Z)(f) \circ j_{A',x'} = j_{A,X(f)(x')} \circ Z(f^{[x']})$$

  for every element x' of $X(A')$.

These assignments are contravariantly functorial from **A** to **Set**:

- Let A be an object of **A**. We have

  $$X(1_A)(x) = 1_{X(A)}(x) = x$$

  for every element x of $X(A)$, and therefore $(1_A)^{[x]} = 1_{(A,x)}$ for every element x of $X(A)$. It follows that

  $$Z((1_A)^{[x]}) = Z(1_{(A,x)}) = 1_{Z(A,x)}$$

  for every element x of $X(A)$.

- Let

  $$f : A \longrightarrow A', \quad g : A' \longrightarrow A''$$

  be two morphisms in **A**. The two morphisms

  $$G_0(Z)(g \circ f), \quad G_0(Z)(f) \circ G_0(Z)(g)$$

  have the same domain and the same codomain. To show that they are the same, it suffices to show that for every element x'' of $X(A'')$, we have

  $$G_0(Z)(g \circ f) \circ j_{A'',x''} = G_0(Z)(f) \circ G_0(Z)(g) \circ j_{A'',x''}.$$

  The two morphisms

  $$g^{[x'']} : (A', X(g)(x'')) \longrightarrow (A'', x'')$$

  and

  $$f^{[X(g)(x'')]} : (A, X(f)(X(g)(x''))) \longrightarrow (A', X(g)(x''))$$

  compose into a morphism

  $$g^{[x'']} \circ f^{[X(g)(x'')]} : (A, X(f)(X(g)(x''))) \longrightarrow (A'', x''). \tag{6.5}$$

  We have

  $$X(f)(X(g)(x'')) = (X(f) \circ X(g))(x'') = X(g \circ f)(x'')$$

  because X is contravariantly functorial. The composite (6.5) is therefore the morphism $g \circ f$ in **A** regarded as a morphism from $(A, X(g \circ f)(x''))$ to (A'', x'') in $\mathbf{E}(X)$. In other words, we have

  $$g^{[x'']} \circ f^{[X(g)(x'')]} = (g \circ f)^{[x'']}.$$

It follows that

$$Z((g \circ f)^{[x'']}) = Z(g^{[x'']} \circ f^{[X(g)(x'')]}) = Z(f^{[X(g)(x'')]}) \circ Z(g^{[x'']})$$

by the contravariant functoriality of Z. It now further follows that

$$\begin{aligned}
& G_0(Z)(f) \circ G_0(Z)(g) \circ j_{A'',\, x''} \\
&= G_0(Z)(f) \circ j_{A',\, X(g)(x'')} \circ Z(g^{[x'']}) \\
&= j_{A,\, X(f)(X(g)(x''))} \circ Z(f^{[X(g)(x'')]}) \circ Z(g^{[x'']}) \\
&= j_{A,\, X(g \circ f)(x'')} \circ Z((g \circ f)^{[x'']}) \\
&= G(Z)(g \circ f) \circ j_{A'',x''} \,,
\end{aligned}$$

which is the equality that we needed to prove.

We have thus proven that $G_0(Z)$ is a contravariant functor from $\mathbf{E}(X)$ to $\mathbf{Set}$.

We now have to construct a natural transformation from $G_0(Z)$ to X. For this, we need to construct for every object A of $\mathbf{A}$ a map

$$G_1(Z)_A : G_0(Z)(A) \longrightarrow X(A)\,,$$

such that for every morphism

$$f : A \longrightarrow A'$$

in $\mathbf{A}$, the following square diagram commutes:

$$\begin{tikzcd}
G_0(Z)(A') \arrow[r, "G_0(Z)(f)"] \arrow[d, "G_1(Z)_{A'}"'] & G_0(Z)(A) \arrow[d, "G_1(Z)_A"] \\
X(A') \arrow[r, "X(f)"] & X(A)
\end{tikzcd} \tag{6.6}$$

We first construct the transformation $G_1(Z)$, and then check its naturality.

- We have $G_0(Z)(A) = \coprod_{x \in X(A)} Z(A, x)$. There hence exists a unique set-theoretic map $G_1(Z)_A$ from $G_0(Z)(A)$ to $X(A)$ such that for every element x of $X(A)$ the composite $G_1(Z)_A \circ j_{A,x}$ is constant with value x.

- The diagram (6.6) commutes because

$$
\begin{aligned}
&G_1(Z)_A \circ G_0(Z)(f) \circ j_{A',x'} \\
&= G_1(Z)_A \circ j_{A,X(f)(x')} \circ Z(f^{[x']}) . \\
&= (\text{constant map with value } X(f)(x')) \circ Z(f^{[x']}) \\
&= (\text{constant map with value } X(f)(x')) \\
&= X(f) \circ (\text{constant map with value } x') \\
&= X(f) \circ G_1(Z)_{A'} \circ j_{A',x'}
\end{aligned}
$$

  for every element x' of $X(A')$, and therefore

$$G_1(Z)_A \circ G_0(Z)(f) = X(f) \circ G_1(Z)_{A'} .$$

We have thus constructed a natural transformation $G_1(Z)$ from $G_0(Z)$ to X.

We have overall constructed the action of G on objects. Next, we will construct the action of G on morphisms. Let

$$\gamma : Z \longrightarrow Z'$$

be a morphism in $[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]$. We need to construct a morphism

$$G(\gamma) : G(Z) \longrightarrow G(Z')$$

in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]/X$. In other words $G(\gamma)$ needs to be a natural transformation from $G_0(Z)$ to $G_0(Z')$ that makes the following diagram commute:

$$
\begin{tikzcd}
G_0(Z) \arrow[rr, Rightarrow, "G(\gamma)"] \arrow[dr, Rightarrow, "G_1(Z)"'] & & G_0(Z') \arrow[dl, Rightarrow, "G_1(Z')"] \\
& X &
\end{tikzcd}
\tag{6.7}
$$

- We have for every object A of $\mathbf{A}$ that

$$G_0(Z)(A) = \coprod_{x \in X(A)} Z(A, x) , \quad G_0(Z')(A) = \coprod_{x \in X(A)} Z'(A, x) .$$

  The morphism γ is a natural transformation from Z to Z', and therefore gives us for every object (A, x) of $\mathbf{E}(X)$ a map $\gamma_{(A,x)}$ from $Z(A, x)$ to $Z'(A, x)$. This allows us to define the desired transformation $G(\gamma)$ via

$$G(\gamma)_A \coloneqq \coprod_{x \in X(A)} \gamma_{(A,x)}$$

  for every object A of $\mathbf{A}$.

- To check the commutativity of the diagram (6.7), we need to check that for object A of $\mathbf{A}$ the following diagram commutes:

$$\begin{tikzcd} G_0(Z)(A) \arrow[rr, "G(\gamma)_A"] \arrow[dr, "G_1(Z)_A"'] & & G_0(Z')(A) \arrow[dl, "G_1(Z')_A"] \\ & X(A) & \end{tikzcd}$$

For this, it suffices to check that for every element x of $X(A)$, we have

$$G_1(Z)_A \circ G(\gamma)_A \circ j^Z_{(A,x)} = G_1(Z)_A \circ j^Z_{(A,x)} \,.$$

This equality hold because

$$\begin{aligned} G_1(Z')_A \circ G(\gamma)_A \circ j^Z_{A,x} &= G_1(Z')_A \circ j^{Z'}_{A,x} \circ \gamma_{(A,x)} \\ &= (\text{constant map with value } x) \circ \gamma_{(A,x)} \\ &= (\text{constant map with value } x) \\ &= G_1(Z)_A \circ j^Z_{A,x} \,. \end{aligned}$$

We have overall constructed an induced morphism $G(\gamma)$ from $G(Z)$ to $G(Z')$. We have to check that this construction is functorial.

- Let Z be an object of $[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]$. We have

$$\begin{aligned} G(1_Z)_A &= \coprod_{x \in X(A)} (1_Z)_{(A,x)} \\ &= \coprod_{x \in X(A)} 1_{Z(A,x)} \\ &= 1_{\coprod_{x \in X(A)} Z(A,x)} \\ &= 1_{G_0(Z)(A)} \\ &= (1_{G_0(Z)})_A \end{aligned}$$

for every object A of $\mathbf{A}$, and therefore

$$G(1_Z) = 1_{G_0(Z)} = 1_{G(Z)} \,.$$

- Let

$$\gamma : G \longrightarrow G' \,, \quad \gamma' : G' \longrightarrow G''$$

be two composable morphisms in $[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]$. We have for every object A of $\mathbf{A}$ the equalities

$$\begin{aligned}
(G(\gamma') \circ G(\gamma))_A &= G(\gamma')_A \circ G(\gamma)_A \\
&= \left(\coprod_{x \in X(A)} \gamma'_{(A,x)} \right) \circ \left(\coprod_{x \in X(A)} \gamma_{(A,x)} \right) \\
&= \coprod_{x \in X(A)} (\gamma'_{(A,x)} \circ \gamma_{(A,x)}) \\
&= \coprod_{x \in X(A)} (\gamma' \circ \gamma)_{(A,x)} \\
&= G(\gamma' \circ \gamma) .
\end{aligned}$$

We have thus proven the functoriality of G.

The general case: the isomorphism $G \circ F \cong 1$

Let (Y, α) be an object of $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]/X$. We have

$$(G \circ F)(Y, \alpha)(A) = G(F(Y, \alpha))(A) = \coprod_{x \in X(A)} F(Y, \alpha)(A, x) = \coprod_{x \in X(A)} \alpha_A^{-1}(x) \tag{6.8}$$

for every object A of $\mathbf{A}$, with α_A being a map from $Y(A)$ to $X(A)$. The set $Y(A)$ is the disjoint union of the preimages $\alpha_A^{-1}(x)$ where x ranges trough $X(A)$. We have therefore a bijection

$$\varepsilon_{(Y,\alpha),A} : (G \circ F)(Y, \alpha)(A) \longrightarrow Y(A)$$

that is given for every element x of $X(A)$ on the x-th summand of (6.8) by the inclusion map from $\alpha_A^{-1}(x)$ to $Y(A)$. We denote this inclusion map by

$$i_{(Y,\alpha),A,x} : \alpha_A^{-1}(x) \longrightarrow Y(A) .$$

These bijections $\varepsilon_{(Y,\alpha),A}$ assemble altogether into a transformation

$$\varepsilon_{(Y,\alpha)} : G_0(F(Y, \alpha)) \longrightarrow Y .$$

Let us check that this transformation is natural. To prove this, let

$$f : A \longrightarrow A'$$

be a morphism in **A**. We need to check the commutativity of the following square diagram:

$$\begin{tikzcd} G_0(F(Y,\alpha))(A') \arrow[r, "G_0(F(Y,\alpha))(f)"] \arrow[d, "\varepsilon_{(Y,\alpha),A'}"'] & G_0(F(Y,\alpha))(A) \arrow[d, "\varepsilon_{(Y,\alpha),A}"] \\ Y(A') \arrow[r, "Y(f)"] & Y(A) \end{tikzcd}$$

This diagram may be rewritten as follows:

$$\begin{tikzcd} \coprod_{x'\in X(A')} \alpha_{A'}^{-1}(x') \arrow[r, "G_0(F(Y,\alpha))(f)"] \arrow[d, "\varepsilon_{(Y,\alpha),A'}"'] & \coprod_{x\in X(A)} \alpha_A^{-1}(x) \arrow[d, "\varepsilon_{(Y,\alpha),A}"] \\ Y(A') \arrow[r, "Y(f)"] & Y(A) \end{tikzcd}$$

To check the commutativity of this square diagram, we need to check that

$$\varepsilon_{(Y,\alpha),A} \circ G_0(F(Y,\alpha))(f) \circ j_{A',x'} = Y(f) \circ \varepsilon_{(Y,\alpha),A'} \circ j_{A',x'}$$

for every element x' of $X(A')$. This equality holds because

$$\begin{aligned} &\varepsilon_{(Y,\alpha),A} \circ G_0(F(Y,\alpha))(f) \circ j_{A',x'} \\ &= \varepsilon_{(Y,\alpha),A} \circ j_{A,X(f)(x')} \circ F(Y,\alpha)(f^{[x']}) \\ &= i_{(Y,\alpha),A,X(f)(x')} \circ F(Y,\alpha)(f^{[x']}) \\ &= i_{(Y,\alpha),A,X(f)(x')} \circ Y(f)\Big|_{\alpha_{A'}^{-1}(x')}^{\alpha_A^{-1}(X(f)(x'))} \\ &= Y(f) \circ i_{(Y,\alpha),A',x'} \\ &= Y(f) \circ \varepsilon_{(Y,\alpha),A'} \circ j_{A',x'} \, . \end{aligned}$$

We have thus constructed a natural transformation $\varepsilon_{(Y,\alpha)}$ from $G_0(F(Y,\alpha))$ to Y. Each component of $\varepsilon_{(Y,\alpha)}$ is bijective, whence $\varepsilon_{(Y,\alpha)}$ is a natural isomorphism. We claim that this natural isomorphism is a morphism from (Y,α) to $G(F(Y,\alpha))$. For this, we need to prove that the following diagram commutes:

$$\begin{tikzcd} G_0(F(Y,\alpha)) \arrow[rr, Rightarrow, "\varepsilon_{(Y,\alpha)}"] \arrow[dr, Rightarrow, "G_1(F(Y,\alpha))"'] & & Y \arrow[dl, Rightarrow, "\alpha"] \\ & X & \end{tikzcd}$$

We hence need to show that for every object A of $\mathbf{A}$, the following diagram commutes:

$$\begin{tikzcd} G_0(F(Y,\alpha))(A) \arrow[rr, "\varepsilon_{(Y,\alpha),A}"] \arrow[dr, "G_1(F(Y,\alpha))_A"'] & & Y(A) \arrow[dl, "\alpha_A"] \\ & X(A) & \end{tikzcd}$$

It suffices to show that

$$\alpha_A \circ \varepsilon_{(Y,\alpha),A} \circ j_{(A,x)} = G_1(F(Y,\alpha))_A \circ j_{(A,x)}$$

for every element x of $X(A)$. This equality holds because

$$\begin{aligned} & \alpha_A \circ \varepsilon_{(Y,\alpha),A} \circ j_{(A,x)} \\ &= \alpha_A \circ i_{(Y,\alpha),A,x} \\ &= \text{constant map with value } x \\ &= G_1(F(Y,\alpha))_A \circ j_{(A,x)} . \end{aligned}$$

We have thus constructed for every object (Y,α) a morphism

$$\varepsilon_{(Y,\alpha)} : \ G(F(Y,\alpha)) \longrightarrow (Y,\alpha)$$

in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]/X$. We have also seen that this morphism is an isomorphism in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$; it is therefore also an isomorphism in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]/X$.

The isomorphism $\varepsilon_{(Y,\alpha)}$ is natural in (Y,α). To see this, we consider a morphism

$$\beta : \ (Y,\alpha) \longrightarrow (Y',\alpha')$$

in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]/X$. We need to show that the following square diagram commutes:

$$\begin{tikzcd} GF(Y,\alpha) \arrow[r, "GF(\beta)"] \arrow[d, "\varepsilon_{(Y,\alpha)}"'] & GF(Y',\alpha') \arrow[d, "\varepsilon_{(Y',\alpha')}"] \\ (Y,\alpha) \arrow[r, "\beta"] & (Y',\alpha') \end{tikzcd}$$

In other words, we need to prove the commutativity of the following square diagram of functors and natural transformations between them:

$$\begin{tikzcd} G_0(F(Y,\alpha)) \arrow[r, Rightarrow, "GF(\beta)"] \arrow[d, Rightarrow, "\varepsilon_{(Y,\alpha)}"'] & G_0(F(Y',\alpha')) \arrow[d, Rightarrow, "\varepsilon_{(Y',\alpha')}"] \\ Y \arrow[r, Rightarrow, "\beta"] & Y' \end{tikzcd}$$

This means that we need to prove for every object A of $\mathbf{A}$ the commutativity of the following square diagram in $\mathbf{Set}$:

$$\begin{tikzcd} G_0(F(Y,\alpha))(A) \arrow[r, "G(F(\beta))_A"] \arrow[d, "\varepsilon_{(Y,\alpha),A}"'] & G_0(F(Y',\alpha'))(A) \arrow[d, "\varepsilon_{(Y',\alpha'),A}"] \\ Y(A) \arrow[r, "\beta_A"] & Y'(A) \end{tikzcd}$$

This diagram may be rewritten as follows:

$$\begin{tikzcd} \coprod_{x\in X(A)} \alpha_A^{-1}(x) \arrow[r, "\coprod_{x\in X(A)} F(\beta)_{(A,x)}"] \arrow[d, "\varepsilon_{(Y,\alpha),A}"'] & \coprod_{x\in X(A)} (\alpha')_A^{-1}(x) \arrow[d, "\varepsilon_{(Y',\alpha'),A}"] \\ Y(A) \arrow[r, "\beta_A"] & Y'(A) \end{tikzcd}$$

The map $F(\beta)_{(A,x)}$ is the restriction of $F(\beta)$ to a map from $\alpha_A^{-1}(x)$ to $(\alpha')_A^{-1}(x)$, whence this diagram commutes.

We have thus constructed a natural isomorphism ε from $G \circ F$ to $1_{[\mathbf{A}^{\mathrm{op}},\mathbf{Set}]/X}$. The existence of this isomorphism shows that

$$G \circ F \cong 1_{[\mathbf{A}^{\mathrm{op}},\mathbf{Set}]/X} \,.$$

The general case: the isomorphism $F \circ G \cong 1$

Let Z be an object of $[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]$. For every object (A, x) of $\mathbf{E}(X)$, the set

$$F(G(Z))(A,x) = G_1(Z)_A^{-1}(x)$$

is precisely the image of $Z(A,x)$ in $G_0(Z)(A) = \coprod_{x'\in X(A)} Z(A,x')$. (Recall that the map $G_1(Z)_A$ has the constant value x on the summand $Z(A,x)$ of $G_0(Z)$.) In other words, we have

$$F(G(Z))(A,x) = \{(x,z) \mid z \in Z(A,x)\}\,.$$

We have therefore a bijection

$$\eta_{Z,(A,x)} : Z(A,x) \longrightarrow F(G(Z))(A,x), \quad z \longmapsto (x,z).$$

The bijection $\eta_{Z,(A,x)}$ is natural in (A,x). To see this, we consider a morphism

$$f : (A,x) \longrightarrow (A',x')$$

in $\mathbf{E}(X)$, and need to show that the following square diagram commutes:

$$\begin{tikzcd}
Z(A',x') \arrow[r, "Z(f)"] \arrow[d, "\eta_{Z,(A',x')}"'] & Z(A,x) \arrow[d, "\eta_{Z,(A,x)}"] \\
F(G(Z))(A',x') \arrow[r, "F(G(Z))(f)"] & F(G(Z))(A,x)
\end{tikzcd}$$

The map $F(G(Z))(f)$ is the restriction of $G_0(Z)(f)$, and $G_0(Z)(f)$ maps the summand $Z(A',x')$ of $G_0(Z)(A')$ into the summand $Z(A,x)$ of $G_0(Z)(A)$ via the map $Z(f)$. We thus find that the above diagram commutes.

We have thus for every object Z of $[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]$ a natural transformation η_Z from Z to $F(G(Z))$, i.e., a morphism from Z to $(F \circ G)(Z)$ in $[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]$. Each component of η_Z is bijective, whence η_Z is an isomorphism.

We now check that η_Z is natural in Z. For this, we need to show that for every morphism

$$\gamma : Z \longrightarrow Z'$$

in $[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]$ the following square diagram commutes:

$$\begin{tikzcd}
Z \arrow[r, "\gamma"] \arrow[d, "\eta_Z"'] & Z' \arrow[d, "\eta_{Z'}"] \\
F(G(Z)) \arrow[r, "F(G(\gamma))"] & F(G(Z'))
\end{tikzcd}$$

This is a diagram consisting of functors and natural transformations, whence we need to show that for every object (A,x) of $\mathbf{E}(X)$ the following square diagram commutes:

$$\begin{tikzcd}
Z(A,x) \arrow[r, "\gamma_{(A,x)}"] \arrow[d, "\eta_{Z,(A,x)}"'] & Z'(A,x) \arrow[d, "\eta_{Z',(A,x)}"] \\
F(G(Z))(A,x) \arrow[r, "F(G(\gamma))_{(A,x)}"] & F(G(Z'))(A,x)
\end{tikzcd}$$

We recall that

$$G_0(Z)(A) = \coprod_{x' \in X(A)} Z(A, x'), \qquad G_0(Z')(A) = \coprod_{x' \in X(A)} Z'(A, x')$$

and that the map $G(\gamma)_A$ from $G_0(Z)(A)$ to $G_0(Z')(A)$ is given by $\coprod_{x' \in X(A)} \gamma_{(A,x')}$. The map $F(G(\gamma))_{(A,x)}$ results from $G_0(Z)_A$ by restriction, and goes from the copy of $Z(A, x)$ in $G_0(Z)(A)$ to the copy of $Z'(A, x)$ in $G_0(Z)(A')$. This means precisely that the above square diagram commutes.

We have thus constructed a natural transformation η from $1_{[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]}$ to $F \circ G$. Each component of η is an isomorphism, whence η is a natural isomorphism. The existence of such an isomorphism shows that

$$1_{[\mathbf{E}(X)^{\mathrm{op}}, \mathbf{Set}]} \cong F \circ G\,.$$

Exercise 6.2.25

(a)

Lemma 6.D. Let $\mathscr{A}$ be a category. Let $\mathbf{I}$ and $\mathbf{I}'$ be two small categories, and let D and D' be diagrams in $\mathscr{A}$ of shapes $\mathbf{I}$ and $\mathbf{I}'$ respectively. Suppose that these diagrams admit colimits $(C, (q_I)_I)$ and $(C', (q'_I)_I)$ respectively. Let

$$F : \mathbf{I} \longrightarrow \mathbf{I}'$$

be a functor and let

$$\alpha : D \Longrightarrow D' \circ F$$

be a natural transformation. Then there exists a unique morphism f from C to C' with

$$f \circ q_I = q'_{F(I)} \circ \alpha_I$$

for every object I of $\mathbf{I}$.

Proof. We have for every object I of $\mathbf{I}$ the morphism

$$f_I : D(I) \xrightarrow{\alpha_I} D'(F(I)) \xrightarrow{q'_I} C'\,.$$

We have for every morphism

$$u : I \longrightarrow J$$

in $\mathbf{I}$ the chain of equalities

$$\begin{aligned} f_J \circ D(u) &= q'_{F(J)} \circ \alpha_J \circ D(u) \\ &= q'_{F(J)} \circ (D' \circ F)(u) \circ \alpha_I \\ &= q'_{F(J)} \circ D'(F(u)) \circ \alpha_I \\ &= q'_{F(I)} \circ \alpha_I \\ &= f_I \,. \end{aligned}$$

It follows from the universal property of the colimit $(C, (q_I)_I)$ that the morphisms f_I assemble into a morphism

$$f \colon C \longrightarrow C'$$

with $f \circ q_I = f_I$ for every object I of $\mathbf{I}$. ∎

Proposition 6.E. Let $\mathscr{A}$ be a category. Let $\mathbf{I}$ and $\mathbf{I}'$ be two small categories, and let D and D' be diagrams in $\mathscr{A}$ of shapes $\mathbf{I}$ and $\mathbf{I}'$ respectively. Suppose that these diagrams admit colimits $(C, (q_I)_I)$ and $(C', (q'_I)_I)$ respectively. Let

$$F \colon \mathbf{I} \longrightarrow \mathbf{I}'$$

be a functor with $D' \circ F = D$. There exists a unique morphism f from C to C' with

$$f \circ q_I = q'_{F(I)}$$

for every object I of $\mathbf{I}$. ∎

Construction of $\mathrm{Lan}_F(X)$ on objects

We have for every object B of $\mathbf{B}$ the diagram $X \circ P_B$ of shape $F \Rightarrow B$ in $\mathscr{S}$. We choose for every object B of $\mathbf{B}$ a colimit of the associated diagram $X \circ P_B$. We denote this colimit by $\mathrm{Lan}_F(X)(B)$, and denote for every object M of the category $F \Rightarrow B$ $(X \circ P_B)(M)$ to $\mathrm{Lan}_F(X)(B)$ by i^B_M.

Any object of $F \Rightarrow B$ is a pair (A, h) consisting of an object A of $\mathbf{A}$ and a morphism h from $F(A)$ to B, and the morphism $i^B_{(A,h)}$ is of the form

$$i^B_{(A,h)} \colon X(A) \longrightarrow \mathrm{Lan}_F(X)(B)\,.$$

That $(\mathrm{Lan}_F(X)(B), (i^B_M)_M)$ is a colimit of the diagram $X \circ P_B$ entails that for every morphism

$$f \colon M \longrightarrow M'$$

in $F \Rightarrow B$, we have the equality

$$i^B_{M'} \circ (X \circ P_B)(f) = i^B_M \,. \tag{6.9}$$

The objects M and M' are of the forms $M = (A, h)$ and (A', h') and f is a morphism from A to A' such that $h' \circ f = h$. We can re-express the equality (6.9) as

$$i^B_{(A',h')} \circ X(f) = i^B_{(A,h)} \,. \tag{6.10}$$

In a more diagrammatic formulation, we have the following:

$$\begin{tikzcd} F(A) \arrow[rr, "F(f)"] \arrow[dr, "h"'] & & F(A') \arrow[dl, "h'"] \\ & B & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X(A) \arrow[rr, "X(f)"] \arrow[dr, "i^B_{(A,h)}"'] & & X(A') \arrow[dl, "i^B_{(A',h')}"] \\ & \mathrm{Lan}_F(X)(B) & \end{tikzcd}$$

Construction of $\mathrm{Lan}_F(X)$ on morphisms

Let

$$g : B \longrightarrow B'$$

be a morphism in $\mathbf{B}$. This morphism induces a functor

$$g_* : (F \Rightarrow B) \longrightarrow (F \Rightarrow B') \,,$$

given by

$$g_*((A, h)) = (A, g \circ h) \,, \quad g_*(f) = f$$

on objects and morphisms respectively. The action of g_* may be depicted as follows:

$$\begin{tikzcd} F(A) \arrow[rr, "F(f)"] \arrow[dr, "h"'] & & F(A') \arrow[dl, "h'"] \\ & B & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(A) \arrow[rr, "F(f)"] \arrow[dr, dashed, "h"'] \arrow[ddr, bend right, "g \circ h"'] & & F(A') \arrow[dl, dashed, "h'"] \arrow[ddl, bend left, "g \circ h'"] \\ & B \arrow[d, dashed, "g"] & \\ & B' & \end{tikzcd}$$

The action of the functor g_* on objects of $F \Rightarrow B$ doesn't change the first entry, whence $P_{B'} \circ g_* = P_B$. The following diagram therefore commutes:

$$\begin{tikzcd}
(F \Rightarrow B) \arrow[dd, "g_*"'] \arrow[dr, "P_B"] \arrow[drr, bend left, "X \circ P_B"] & & \\
& \mathbf{A} \arrow[r, "X"] & \mathcal{S} \\
(F \Rightarrow B') \arrow[ur, "P_{B'}"'] \arrow[urr, bend right, "X \circ P_{B'}"'] & &
\end{tikzcd}$$

This entails that the outer triangle

$$\begin{tikzcd}
(F \Rightarrow B) \arrow[dd, "g_*"'] \arrow[dr, "X \circ P_B"] & \\
& \mathcal{S} \\
(F \Rightarrow B') \arrow[ur, "X \circ P_{B'}"'] &
\end{tikzcd}$$

commutes. It follows from Proposition 6.E that the functor g_* induces a morphism

$$\mathrm{Lan}_F(X)(g) : \ \mathrm{Lan}_F(X)(B) \longrightarrow \mathrm{Lan}_F(X)(B')\,.$$

This induced morphism is unique with the property

$$\mathrm{Lan}_F(X)(g) \circ i^B_M = i^{B'}_{g_*(M)} \tag{6.11}$$

for every object M of $F \Rightarrow B$.

Checking the functoriality of $\mathrm{Lan}_F(X)$

Let us shows that the assignment $\mathrm{Lan}_F(X)$ from $\mathbf{B}$ to $\mathcal{S}$ is functorial.

- Let B be an object of $\mathbf{B}$. We need to check that $\mathrm{Lan}_F(X)(1_B) = 1_{\mathrm{Lan}_F(X)(B)}$. For this, it suffices to check that

  $$\mathrm{Lan}_F(X)(1_B) \circ i^B_M = 1_{\mathrm{Lan}_F(X)(B)} \circ i^B_M$$

  for every object M of $F \Rightarrow B$. This equality holds because the functor $(1_B)_*$ from $F \Rightarrow B$ to $F \Rightarrow B$ is the identity functor, and therefore

  $$\mathrm{Lan}_F(X)(1_B) \circ i^B_M = i^B_{(1_B)_*(M)} = i^B_M = 1_{\mathrm{Lan}_F(X)(1_B)} \circ i^B_M\,.$$

- Let
$$g : B \longrightarrow B', \quad g' : B' \longrightarrow B''$$
be two composable morphisms in $\mathbf{B}$. The induced functors
$$g_* : (F \Rightarrow B) \longrightarrow (F \Rightarrow B'), \quad (g')_* : (F \Rightarrow B') \longrightarrow (F \Rightarrow B''),$$
$$(g' \circ g)_* : (F \Rightarrow B) \longrightarrow (F \Rightarrow B'')$$
satisfy the identity $g'_* \circ g_* = (g' \circ g)_*$. It follows that
$$\begin{aligned} \mathrm{Lan}_F(X)(g') \circ \mathrm{Lan}_F(X)(g) \circ i^B_M = \mathrm{Lan}_F(X)(g') \circ i^{B'}_{g_*(M)} &= i^{B''}_{g'_*(g_*(M))} \\ &= i^{B''}_{(g'_* \circ g_*)(M)} \\ &= i^{B''}_{(g' \circ g)_*(M)} \end{aligned}$$
for every object M of $F \Rightarrow B$. The composite $\mathrm{Lan}_F(X)(g') \circ \mathrm{Lan}_F(X)(g)$ therefore satisfies the defining property of the morphism $\mathrm{Lan}_F(X)(g' \circ g)$, which tells us that
$$\mathrm{Lan}_F(X)(g' \circ g) = \mathrm{Lan}_F(X)(g') \circ \mathrm{Lan}_F(X)(g).$$

We have thus proven the functoriality of $\mathrm{Lan}_F(X)$ from $\mathbf{B}$ to $\mathcal{S}$.

The natural transformation $\eta : X \Rightarrow \mathrm{Lan}_F(X) \circ F$

We note that every object A of $\mathbf{A}$ results in an object $F(A)$ of $\mathbf{B}$, which furthermore results in the object $(A, 1_{F(A)})$ of the category $F \Rightarrow F(A)$. We have therefore for every object A of $\mathbf{A}$ the morphism
$$i^{F(A)}_{(A, 1_{F(A)})} : X(A) \longrightarrow \mathrm{Lan}_F(X)(F(A)).$$
Let us abbreviate this morphism by η_A. It seems reasonable to assume that this transformation η from X to $\mathrm{Lan}_F(X) \circ F$ is natural. To check this naturality, we need to verify that for every morphism
$$f : A \longrightarrow A'$$
in $\mathbf{A}$, the following square diagram commutes:
$$\begin{tikzcd} X(A) \arrow[r, "X(f)"] \arrow[d, "\eta_A"'] & X(A') \arrow[d, "\eta_{A'}"] \\ \mathrm{Lan}_F(X)(F(A)) \arrow[r, "\mathrm{Lan}_F(X)(F(f))"] & \mathrm{Lan}_F(X)(F(A')) \end{tikzcd}$$

The morphism $F(f)$ goes from $F(A)$ to $F(A')$. By construction of the morphism $\mathrm{Lan}_F(X)(F(f))$ (see (6.11)), we have therefore the chain of equalities

$$\mathrm{Lan}_F(X)(F(f)) \circ \eta_A = \mathrm{Lan}_F(X)(F(f)) \circ i^{F(A)}_{(A,1_{F(A)})} = i^{F(A')}_{F(f)_*((A,1_{F(A)}))} = i^{F(A')}_{(A,F(f))}\,.$$

We have on the other hand the equality

$$\eta_{A'} \circ X(f) = i^{F(A')}_{(A',1_{F(A')})} \circ X(f)\,.$$

The desired commutativity is therefore equivalent to the equality

$$i^{F(A')}_{(A',1_{F(A')})} \circ X(f) = i^{F(A')}_{(A,F(f))}\,.$$

This equality holds because f is a morphism from $(A, F(f))$ to $(A', 1_{F(A')})$ in the category $F \Rightarrow F(A')$ (see (6.10)).

From $\mathrm{Lan}_F(X) \Rightarrow Y$ to $X \Rightarrow Y \circ F$

Let from now Y be any functor from $\mathbf{B}$ to $\mathcal{S}$. If β is a natural transformation from $\mathrm{Lan}_F(X)$ to Y, then βF is a natural transformation from $\mathrm{Lan}_F(X) \circ F$ to $Y \circ F$, whence

$$\beta F \circ \eta$$

is a natural transformation from X to $Y \circ F$.

From $X \Rightarrow Y \circ F$ to $\mathrm{Lan}_F(X) \Rightarrow Y$

Let conversely α be a natural transformation from X to $Y \circ F$. Let B be an object of $\mathbf{B}$. For every object (A, h) of $F \Rightarrow B$, we have the resulting morphism

$$\tilde{\beta}_{(A,h)} :\ X(A) \xrightarrow{\alpha_A} Y(F(A)) \xrightarrow{Y(h)} Y(B)\,.$$

This is a morphism from the object $X(A) = (X \circ P_B)((A, h))$ to the object $Y(B)$.

We claim that these morphisms, where (A, h) ranges through the category $F \Rightarrow B$, form a cocone for the diagram $X \circ P_B$. To prove this claim, we consider a morphism

$$f :\ (A, h) \longrightarrow (A', h')$$

in $F \Rightarrow B$, and need to show that

$$\tilde{\beta}_{(A',h')} \circ (X \circ P_B)(f) = \tilde{\beta}_{(A,h)}\,.$$

This equality holds because

$$\begin{aligned}\tilde{\beta}_{(A',h')} \circ (X \circ P_B)(f) &= Y(h') \circ \alpha_{A'} \circ X(f) \\ &= Y(h') \circ (Y \circ F)(f) \circ \alpha_A \\ &= Y(h') \circ Y(F(f)) \circ \alpha_A \\ &= Y(h' \circ F(f)) \circ \alpha_A \\ &= Y(h) \circ \alpha_A \\ &= \tilde{\beta}_{(A,h)} \,.\end{aligned}$$

We know that $(\mathrm{Lan}_F(X), (i^B_M)_M)$ is a colimit of the diagram $X \circ P_B$, by construction. It follows that the morphisms $\tilde{\beta}_M$, where M ranges through $F \Rightarrow B$, induce a morphism β_B from $\mathrm{Lan}_F(X)(B)$ to $Y(B)$, which is unique with the property that

$$\beta_B \circ i^B_M = \tilde{\beta}_M$$

for every object M of $F \Rightarrow B$.

These morphisms β_B, where B ranges through the objects of $\mathbf{B}$, assemble into a transformation β from $\mathrm{Lan}_F(X)$ to Y. Let us check that the transformation β is natural. To show this, we need to check that for every morphism

$$g : B \longrightarrow B'$$

in $\mathbf{B}$ the following square diagram commutes:

$$\begin{tikzcd} \mathrm{Lan}_F(X)(B) \arrow[r, "\mathrm{Lan}_F(X)(g)"] \arrow[d, "\beta_B"'] & \mathrm{Lan}_F(X)(B') \arrow[d, "\beta_{B'}"] \\ Y(B) \arrow[r, "Y(g)"] & Y(B') \end{tikzcd}$$

In other words, we need to prove the equality

$$\beta_{B'} \circ \mathrm{Lan}_F(X)(g) = Y(g) \circ \beta_B \,.$$

Both sides of this desired equation are morphisms with domain $\mathrm{Lan}_F(X)(B)$. We thus need to show that

$$\beta_{B'} \circ \mathrm{Lan}_F(X)(g) \circ i^B_M = Y(g) \circ \beta_B \circ i^B_M$$

for every object M of $F \Rightarrow B$. Such an object M is of the form (A, h), and we have the chain of equalities

$$\begin{aligned}\beta_{B'} \circ \mathrm{Lan}_F(X)(g) \circ i^B_{(A,h)} &= \beta_{B'} \circ i^{B'}_{g_*((A,h))} \\ &= \beta_{B'} \circ i^{B'}_{(A,g\circ h)} \\ &= \tilde{\beta}_{(A,g\circ h)} \\ &= Y(g \circ h) \circ \alpha_A \\ &= Y(g) \circ Y(h) \circ \alpha_A \\ &= Y(g) \circ \tilde{\beta}_{(A,h)} \\ &= Y(g) \circ \beta_B \circ i^B_{(A,h)} \,.\end{aligned}$$

We have thus constructed a natural transformation β from $\mathrm{Lan}_F(X)$ to Y.

Correspondence between $X \Rightarrow Y \circ F$ and $\mathrm{Lan}_F(X) \Rightarrow Y$

The above two constructions are mutually inverse.

Let first α be a natural transformation from X to $Y \circ F$. Let β be the resulting natural transformation from $\mathrm{Lan}_F(X)$ to Y, which is uniquely determined by

$$\beta_B \circ i^B_{(A,h)} = Y(h) \circ \alpha_A$$

for every object B of $\mathbf{B}$ and every object (A, h) of $F \Rightarrow B$. Let α' be the natural transformation from X to $Y \circ F$ resulting from β, given by

$$\alpha' = \beta F \circ \eta \,.$$

For every object A of $\mathbf{A}$ we have the chain of equalities

$$\begin{aligned}\alpha'_A = (\beta F \circ \eta)_A = \beta_{F(A)} \circ \eta_A = \beta_{F(A)} \circ i^{F(A)}_{(A,1_{F(A)})} = Y(1_{F(A)}) \circ \alpha_A &= 1_{Y(F(A))} \circ \alpha_A \\ &= \alpha_A \,.\end{aligned}$$

This shows that $\alpha' = \alpha$.

Let conversely β be a natural transformation from $\mathrm{Lan}_F(X)$ to Y. Let α be the resulting natural transformation from X to $Y \circ F$ given by

$$\alpha = \beta F \circ \eta \,.$$

Let β' be the resulting natural transformation from $\mathrm{Lan}_F(X)$ to Y, which is uniquely determined by

$$\beta'_B \circ i^B_{(A,h)} = Y(h) \circ \alpha_A$$

for every object B of $\mathbf{B}$ and every object (A, h) of $F \Rightarrow B$. We have the chain of equalities

$$\begin{aligned}
\beta'_B \circ i^B_{(A,h)} &= Y(h) \circ \alpha_A \\
&= Y(h) \circ (\beta F \circ \eta)_A \\
&= Y(h) \circ \beta_{F(A)} \circ \eta_A \\
&= Y(h) \circ \beta_{F(A)} \circ i^{F(A)}_{(A,1_{F(A)})} \\
&= \beta_B \circ \mathrm{Lan}_F(X)(h) \circ i^{F(A)}_{(A,1_{F(A)})} \\
&= \beta_B \circ i^B_{h_*((A,1_{F(A)}))} \\
&= \beta_B \circ i^B_{(A,h\circ 1_{F(A)})} \\
&= \beta_B \circ i^B_{(A,h)}
\end{aligned}$$

for every object B of $\mathbf{B}$ and every object (A, h) of $F \Rightarrow B$, therefore

$$\beta'_B = \beta_B$$

for every object B of $\mathbf{B}$, and thus $\beta' = \beta$.

This shows that the two constructions are indeed mutually inverse.

(b)

We start off by extending Lan_F to a fully-fledged functor from $[\mathbf{A}, \mathcal{S}]$ to $[\mathbf{B}, \mathcal{S}]$. We then show that Lan_F is left adjoint to the functor F^* from $[\mathbf{B}, \mathcal{S}]$ to $[\mathbf{A}, \mathcal{S}]$.

Action of Lan_F on morphisms

So far, we have only explained how Lan_F acts on objects. Let us construct an action of Lan_F on morphisms too.

Let X and X' be two functors from $\mathbf{A}$ to $\mathcal{S}$ and let

$$\gamma : X \Longrightarrow X'$$

be a natural transformation, i.e., a morphism in $[\mathbf{A}, \mathcal{S}]$. We have for every object B of $\mathbf{B}$ the induced natural transformation

$$\gamma P_B : X \circ P_B \Longrightarrow X' \circ P_B \,.$$

The two functors $X \circ P_B$ and $X' \circ P_B$ are diagrams of the same shape $F \Rightarrow B$ in $\mathcal{S}$, with colimits $\mathrm{Lan}_F(X)(B)$ and $\mathrm{Lan}_F(X')(B)$ respectively. The natural transformation γP_B therefore induces a morphism

$$\mathrm{Lan}_F(\gamma)_B : \mathrm{Lan}_F(X)(B) \longrightarrow \mathrm{Lan}_F(X')(B)$$

that is unique with the property

$$\mathrm{Lan}_F(\gamma)_B \circ i_M^{X,B} = i_M^{X',B} \circ (\gamma P_B)_M$$

for every object M of $F \Rightarrow B$. (We use now the slightly more expressive notations $i_M^{X,B}$ and $i_M^{X',B}$ instead of just i_M^B, to track which functor we are dealing with.) This equation can somewhat more explicitly be rewritten as

$$\mathrm{Lan}_F(\gamma)_B \circ i_{(A,h)}^{X,B} = i_{(A,h)}^{X',B} \circ \gamma_A$$

for every object (A, h) of $F \Rightarrow B$.

The morphisms $\mathrm{Lan}_F(\gamma)_B$, where B ranges through the objects of $\mathbf{B}$, assemble into a transformation $\mathrm{Lan}_F(\gamma)$ from $\mathrm{Lan}_F(X)$ to $\mathrm{Lan}_F(X')$. Let us check that this transformation $\mathrm{Lan}_F(\gamma)$ is natural. To this end, we have to check that for every morphism

$$g : X \longrightarrow X'$$

in $\mathbf{B}$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Lan}_F(X)(B) \arrow[r, "\mathrm{Lan}_F(X)(g)"] \arrow[d, "\mathrm{Lan}_F(\gamma)_B"'] & \mathrm{Lan}_F(X)(B') \arrow[d, "\mathrm{Lan}_F(\gamma)_{B'}"] \\
\mathrm{Lan}_F(X')(B) \arrow[r, "\mathrm{Lan}_F(X')(g)"] & \mathrm{Lan}_F(X')(B')
\end{tikzcd}$$

We thus need to prove the equality

$$\mathrm{Lan}_F(\gamma)_{B'} \circ \mathrm{Lan}_F(X)(g) = \mathrm{Lan}_F(X')(g) \circ \mathrm{Lan}_F(\gamma)_B \,.$$

Both sides of this desired equality have the object $\mathrm{Lan}_F(X)(B)$ as its domain, whence we need to show check that

$$\mathrm{Lan}_F(\gamma)_{B'} \circ \mathrm{Lan}_F(X)(g) \circ i_M^{X,B} = \mathrm{Lan}_F(X')(g) \circ \mathrm{Lan}_F(\gamma)_B \circ i_M^{X,B} .$$

for every object M of $F \Rightarrow B$. Each such object M is of the form (A, h), and we observe the chain of equalities

$$\begin{aligned}
&\mathrm{Lan}_F(\gamma)_{B'} \circ \mathrm{Lan}_F(X)(g) \circ i_{(A,h)}^{X,B} \\
&= \mathrm{Lan}_F(\gamma)_{B'} \circ i_{(A,g\circ h)}^{X,B'} \\
&= i_{(A,g\circ h)}^{X',B'} \circ \gamma_A \\
&= \mathrm{Lan}_F(X')(g) \circ i_{(A,h)}^{X',B} \circ \gamma_A \\
&= \mathrm{Lan}_F(X')(g) \circ \mathrm{Lan}_F(\gamma)_B \circ i_{(A,h)}^{X,B} .
\end{aligned}$$

We have thus shown the naturality of $\mathrm{Lan}_F(\gamma)$, and therefore constructed the action of Lan_F on morphisms of $[\mathbf{A}, \mathscr{S}]$.

Functoriality of Lan_F

Let us check that the assignment Lan_F is functorial.

- Let X be a functor from $\mathbf{A}$ to $\mathscr{S}$. We have

  $$(1_{\mathrm{Lan}_F(X)})_B \circ i_{(A,h)}^{X,B} = 1_{\mathrm{Lan}_F(X)(B)} \circ i_{(A,h)}^{X,B} = i_{(A,h)}^{X,B} = i_{(A,h)}^{X,B} \circ 1_{X(A)} = i_{(A,h)}^{X,B} \circ (1_X)_A$$

  for every object B of $\mathbf{B}$ and every object (A, h) of $F \Rightarrow B$. This shows that the natural transformation $1_{\mathrm{Lan}_F(X)}$ satisfies the defining property of $\mathrm{Lan}_F(1_X)$, whence

  $$\mathrm{Lan}_F(1_X) = 1_{\mathrm{Lan}_F(X)} .$$

- Let X, X' and X'' be functors from $\mathbf{A}$ to $\mathscr{S}$ and let

  $$\gamma : X \Longrightarrow X' , \quad \gamma' : X' \Longrightarrow X''$$

  be two composable natural transformations. We have thi chain of equalities

  $$\begin{aligned}
  (\mathrm{Lan}_F(\gamma') \circ \mathrm{Lan}_F(\gamma))_B \circ i_{(A,h)}^{X,B} &= \mathrm{Lan}_F(\gamma')_B \circ \mathrm{Lan}_F(\gamma)_B \circ i_{(A,h)}^{X,B} \\
  &= \mathrm{Lan}_F(\gamma)'_B \circ i_{(A,h)}^{X',B} \circ \gamma_A \\
  &= i_{(A,h)}^{X'',B} \circ \gamma'_A \circ \gamma_A \\
  &= i_{(A,h)}^{X'',B} \circ (\gamma' \circ \gamma)_A \\
  &= \mathrm{Lan}_F(\gamma' \circ \gamma)_B \circ i_{(A,h)}^{X,B}
  \end{aligned}$$

for every object B of $\mathbf{B}$ and every object (A, h) of $F \Rightarrow B$. This shows that the composite $\mathrm{Lan}_F(\gamma') \circ \mathrm{Lan}_F(\gamma)$ satisfies the defining property of the natural transformation $\mathrm{Lan}_F(\gamma' \circ \gamma)$, whence

$$\mathrm{Lan}_F(\gamma' \circ \gamma) = \mathrm{Lan}_F(\gamma') \circ \mathrm{Lan}_F(\gamma) .$$

We have thus shown the functoriality of Lan_F from $[\mathbf{A}, \mathcal{S}]$ to $[\mathbf{B}, \mathcal{S}]$.

Adjointness of Lan_F and F^*

For every functor X from $\mathbf{A}$ to $\mathcal{S}$ we have constructed in part (a) of this exercise a natural transformation

$$\eta_X : X \Longrightarrow \mathrm{Lan}_F(X) \circ F ,$$

so that for every functor Y from $\mathbf{B}$ to $\mathcal{S}$, the map

$$[\mathbf{B}, \mathcal{S}](\mathrm{Lan}_F(X), Y) \longrightarrow [\mathbf{A}, \mathcal{S}](X, Y \circ F) , \quad \beta \longmapsto \beta F \circ \eta_X \tag{6.12}$$

is bijective.

The natural transformations η_X, where X ranges through the objects of the category $[\mathbf{A}, \mathcal{S}]$, are morphisms in $[\mathbf{A}, \mathcal{S}]$. These morphisms assemble into a transformation η from $1_{[\mathbf{A}, \mathcal{S}]}$ to $\mathrm{Lan}_F(-) \circ F = F^* \circ \mathrm{Lan}_F$.

To see this, we have to check that for every two functors X and X' from $\mathbf{A}$ to $\mathcal{S}$ and every natural transformation

$$\gamma : X \longrightarrow X'$$

the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, Rightarrow, "\gamma"] \arrow[d, Rightarrow, "\eta_X"'] & X' \arrow[d, Rightarrow, "\eta_{X'}"] \\ \mathrm{Lan}_F(X) \circ F \arrow[r, Rightarrow, "\mathrm{Lan}_F(\gamma)F"] & \mathrm{Lan}_F(X') \circ F \end{tikzcd}$$

This diagram takes place inside a functor category, whence it's commutativity can be checked pointwise. We thus have to show that for every object A of $\mathbf{A}$ the following diagram commutes:

$$\begin{tikzcd} X(A) \arrow[r, "\gamma_A"] \arrow[d, "(\eta_X)_A"'] & X'(A) \arrow[d, "(\eta_{X'})_A"] \\ \mathrm{Lan}_F(X)(F(A)) \arrow[r, "(\mathrm{Lan}_F(\gamma)F)_A"] & \mathrm{Lan}_F(X')(F(A)) \end{tikzcd}$$

This commutativity hold because

$$
\begin{aligned}
(\mathrm{Lan}_F(\gamma)F)_A \circ (\eta_X)_A &= \mathrm{Lan}_F(\gamma)_{F(A)} \circ i^{X,F(X)}_{(A,1_{F(X)})} \\
&= i^{X',F(X)}_{(A,1_{F(X)})} \circ \gamma_A \\
&= (\eta_{X'})_A \circ \gamma_A \,.
\end{aligned}
$$

To summarize our findings: We have overall constructed a natural transformation η from $1_{[\mathbf{A},\mathcal{S}]}$ to $F^* \circ \mathrm{Lan}_F$. The bijectivity of (6.12) tells us that for every object X of $[\mathbf{A}, \mathcal{S}]$, the morphism η_X is an initial object of the comma category $X \Rightarrow F^*$.

It follows from Theorem 2.3.6 and its proof that Lan_F is left adjoint to F^*, with η being the unit of one such adjunction.

(c)

Proposition 6.F. Let $\mathbf{A}$ and $\mathbf{B}$ be two small categories and let F be a functor from $\mathbf{A}$ to $\mathbf{B}$. The induced functor

$$F_* : [\mathbf{B}, \mathbf{Set}] \longrightarrow [\mathbf{A}, \mathbf{Set}]$$

admits both a left adjoint and a right adjoint.

Proof. The category $\mathbf{Set}$ admits both small limits and small colimits. The assertion therefore follows from part (b) of this exercise and its dual. More explicitly, the left adjoint of F^* is given by the left Kan extension Lan_F, and the right adjoint of F^* is given by the right Kan extension Ran_F. ∎

Let G and H be two groups and let φ be a homomorphism of groups from G to H. We may regard G and H as small one-object categories $\mathbf{G}$ and $\mathbf{H}$ respectively, and the homomorphism φ as a functor Φ from G to H. The functor categories $[\mathbf{G}, \mathbf{Set}]$ and $[\mathbf{H}, \mathbf{Set}]$ are isomorphic to the categories of G-sets and H-sets respectively, and the functor

$$\Phi^* : [\mathbf{H}, \mathbf{Set}] \longrightarrow [\mathbf{G}, \mathbf{Set}]$$

corresponds to the pull-back functor

$$\varphi^* : H\text{-}\mathbf{Set} \longrightarrow G\text{-}\mathbf{Set}$$

induced by φ. We know from Proposition 6.F that the functor Φ^* admits both a left adjoint and a right adjoint. As a consequence, the functor φ^* admits both a left adjoint and a right adjoint.

For the trivial group 1, the category 1-**Set** is in turn isomorphic to **Set**. We therefore find the following:

- By choosing the group H as trivial, we arrive at the functor

  $$\mathbf{Set} \longrightarrow G\text{-}\mathbf{Set}$$

  that regards every set as a trivial G-set. This functor admits both a left adjoint and a right adjoint.
- By choosing the group G as trivial, we see that the forgetful functor

  $$H\text{-}\mathbf{Set} \longrightarrow \mathbf{Set}$$

  admits both a left adjoint and a right adjoint.

6.3 Interactions between adjoints functors and limits

Exercise 6.3.21

(a)

If U were to admit a right adjoint, then it would preserve colimits, and thus in particular initial objects. This would mean that for the trivial group 1, the set $U(1)$ would be initial in **Set**. But the initial object of **Set** is the empty set, and $U(1)$ is non-empty.

(b)

The functor I regards every set as an indiscrete category in the following sense: gives a set A, the category $I(A)$ has the set A as its class of objects, and for every two elements a and a' of A, there exists a unique morphism from a to a' in $I(A)$. Given two non-empty sets A and B, there exist no morphisms between $I(A)$ and $I(B)$ inside $I(A) + I(B)$. The categories $I(A + B)$ and $I(A) + I(B)$ are therefore not isomorphic. This tells us that the functor I

does not preserve binary coproducts, and therefore cannot be a left adjoint. In other words, I doesn't have a right adjoint.

The functor C assigns to each small category its set of connected components. More explicitly, for every small category $\mathbf{A}$, the set $C(\mathbf{A})$ is the quotient set $\mathrm{Ob}(\mathbf{A})/{\sim}$, where $\sim$ is the equivalence relation generated by

$$A \sim B \quad \text{if} \quad \text{there exists a morphism } A \to B\,.$$

Let us show that the functor C does not preserve pullbacks.

Given a small category $\mathbf{A}$ and subcategories $\mathbf{B}$ and $\mathbf{B}'$ of $\mathbf{A}$, their intersection $\mathbf{B} \cap \mathbf{B}'$ is again a subcategory of $\mathbf{A}$. We have the following commutative diagram of small categories and inclusion functors:

$$\begin{tikzcd} \mathbf{B} \cap \mathbf{B}' \arrow[r] \arrow[d] & \mathbf{B}' \arrow[d] \\ \mathbf{B} \arrow[r] & \mathbf{A} \end{tikzcd}$$

This diagram is a pullback diagram. So if C were to preserve pullbacks, then it would follow that the induced diagram

$$\begin{tikzcd} C(\mathbf{B} \cap \mathbf{B}') \arrow[r] \arrow[d] & C(\mathbf{B}') \arrow[d] \\ C(\mathbf{B}) \arrow[r] & C(\mathbf{A}) \end{tikzcd}$$

would again a pullback diagram. But this is not always the case!

To see this we consider the category $\mathbf{1}$ that consists of two objects 0 and 1 and precisely one non-identity morphism, which goes from 0 to 1. Let $\mathbf{0}_0$ and $\mathbf{0}_1$ be the subcategories of $\mathbf{1}$ consisting of the single objects 0 and 1 respectively. We have in $\mathbf{Cat}$ the following pushout diagram, where all arrows are inclusions functors:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \mathbf{0}_1 \arrow[d] \\ \mathbf{0}_0 \arrow[r] & \mathbf{1} \end{tikzcd}$$

By applying the functor C to this diagram, we get the following commutative diagram:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \{*\} \arrow[d] \\ \{*\} \arrow[r] & \{*\} \end{tikzcd}$$

But this is not a pullback diagram!

(c)

The functor ∇ from **Set** to $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ is defined on objects by $\nabla(B)(X) = B$ and $\nabla(B)(U) = \{*\}$ for every proper open subset U of X.

Suppose that the space X is non-empty. (Otherwise, the functor ∇ is an isomorphism, and therefore admits both a left adjoint and a right adjoint.) This assumption ensures that the space X admits a proper open subset, namely the empty subset $\emptyset$. It then follows for any two sets B and B' that

$$\begin{aligned} (\nabla(B) + \nabla(B'))(\emptyset) &= \nabla(B)(\emptyset) + \nabla(B')(\emptyset) \\ &= \{*\} + \{*\} \\ &\ncong \{*\} \\ &= \nabla(B + B')(\emptyset)\,. \end{aligned}$$

This non-isomorphism shows that the functor ∇ does not preserve binary coproducts, and does therefore not admit a right adjoint.

The functor Λ is defined dually to the functor ∇. Given a set B, the functor $\Lambda(B)$ is given by $\Lambda(B)(\emptyset) = B$ and $\Lambda(B)(U) = \emptyset$ for every non-empty open subset of X. Suppose again that the space X is non-empty. (Otherwise, the functor Λ is an isomorphism, and therefore admits both a left adjoint and a right adjoint.) This ensures that the space X admits a non-empty open subset, namely X itself. We recall that the terminal object of the presheaf category $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ is the presheaf with constant value $\{*\}$. But for every set B, we have $\Lambda(B)(X) = \emptyset \ncong \{*\}$. We thus find that for every set B, the presheaf $\Lambda(B)$ is not terminal in $[\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$. This observation entails that the functor Λ does not preserve terminal objects, and therefore does not admit a left adjoint.

Exercise 6.3.22

(a)

The implication (A) $\implies$ (R)

Suppose that the functor U admits a left adjoint F. We have for every object A of $\mathscr{A}$ the chain of isomorphisms

$$U(A) \cong \mathbf{Set}(\{*\}, U(A)) \cong \mathscr{A}(F(\{*\}), A)\,,$$

which is natural in A. This tells us that the functor U is represented by the object $F(\{*\})$ of $\mathscr{A}$.

The implication (R) $\implies$ (L)

This follows from Proposition 6.2.2.

(b)

Suppose that the functor U is represented by an object A of $\mathscr{A}$. We may assume for simplicity that $U = \mathscr{A}(A, -)$.

The desired left adjoint F of U needs to preserve coproducts, and therefore need to satisfy

$$F(B) \cong F\left(\sum_{b \in B} \{b\}\right) \cong \sum_{b \in B} F(\{b\}) \cong \sum_{b \in B} F(\{*\})$$

for every set B. We also find that

$$\mathscr{A}(F(\{*\}), A') \cong \mathbf{Set}(\{*\}, U(A')) \cong U(A') = \mathscr{A}(A, A')$$

for every object A' of $\mathscr{A}$, natural in A', and therefore

$$F(\{*\}) \cong A$$

by Yoneda's lemma.

Motivated by this thought experiment on the behaviour of F, we define the functor F from $\mathbf{Set}$ to $\mathscr{A}$ as follows.

- For every set B let $F(B)$ be the B-fold sum of A, i.e.,

$$F(B) \coloneqq \sum_{b \in B} A \,.$$

  We further denote for every element b of B the canonical morphism belonging to the b-th summand of $F(B)$ by i^B_b. This is a morphism from A to $F(B)$.

- For every map

$$g \colon B \longrightarrow B' \,,$$

  let $F(g)$ be the unique morphism from $F(B)$ to $F(B')$ given by

$$F(g) \circ i^B_b = i^{B'}_{g(b)}$$

  for every element b of B. (Intuitively speaking, the morphism $F(g)$ maps the b-th summand of $F(B)$ into the $g(b)$-th summand of $F(B')$ – both these summands are A – via the identity morphism of A.)

We have for every set B and every object A' of $\mathscr{A}$ the chain of bijections

$$\begin{aligned} \mathscr{A}(F(B), A') &\cong \mathscr{A}\biggl(\sum_{b \in B} A, A'\biggr) \\ &\cong \prod_{b \in B} \mathscr{A}(A, A') \\ &= \prod_{b \in B} U(A') \\ &\cong \mathbf{Set}(B, U(A')) \,. \end{aligned}$$

The overall bijection is given by

$$\alpha_{B,A'} \colon \mathscr{A}(F(B), A') \longrightarrow \mathbf{Set}(B, U(A')) \,, \quad f \longmapsto [b \longmapsto f \circ i^B_b] \,.$$ [6]

These bijections are natural in both B and A'. Let us check this.

[6] If we only assume that U is isomorphic to $\mathscr{A}(A, -)$, then this formula becomes slightly messier. Indeed, we need to fix one such isomorphism, and then include its components in this formula. The author doesn't want to bother with this additional notational ballast, and therefore choose U to be $\mathscr{A}(A, -)$.

- Let

  $$g : B \longrightarrow B'$$

  be a map of sets. We need to show that the square diagram

  $$\begin{tikzcd} \mathscr{A}(F(B'), A') \arrow[r, "\alpha_{B',A'}"] \arrow[d, "F(g)^*"'] & \mathbf{Set}(B', U(A')) \arrow[d, "g^*"] \\ \mathscr{A}(F(B), A') \arrow[r, "\alpha_{B,A'}"] & \mathbf{Set}(B, U(A')) \end{tikzcd}$$

  commutes. This commutativity holds because we have the chain of equalities

  $$\begin{aligned} g^*(\alpha_{B',A'}(f))(b) &= (\alpha_{B',A'}(f) \circ g)(b) \\ &= \alpha_{B',A'}(f)(g(b)) \\ &= f \circ i^{B'}_{g(b)} \\ &= f \circ F(g) \circ i^B_b \\ &= \alpha_{B,A'}(f \circ F(g))(b) \\ &= \alpha_{B,A'}(F(g)^*(f))(b) \end{aligned}$$

  for every element f of $\mathscr{A}(F(B'), A')$ and every element b of B.

- Let

  $$h : A' \longrightarrow A''$$

  be a morphism in $\mathscr{A}$. We need to show that the square diagram

  $$\begin{tikzcd} \mathscr{A}(F(B), A') \arrow[r, "\alpha_{B,A'}"] \arrow[d, "h_*"'] & \mathbf{Set}(B, U(A')) \arrow[d, "U(h)_*"] \\ \mathscr{A}(F(B), A'') \arrow[r, "\alpha_{B,A''}"] & \mathbf{Set}(B, U(A'')) \end{tikzcd}$$

  commutes. This commutativity holds because we have the chain of equalities

$$\begin{aligned} U(h)_*(\alpha_{B,A'}(f))(b) &= (U(h) \circ \alpha_{B,A'}(f))(b) \\ &= U(h)(\alpha_{B,A'}(f)(b)) \\ &= U(h)(f \circ i_b^B) \\ &= h_*(f \circ i_b^B) \\ &= h \circ f \circ i_b^B \\ &= h_*(f) \circ i_b^B \\ &= \alpha_{B,A''}(h_*(f)) \end{aligned}$$

for every element f of $\mathscr{A}(F(B), A')$ and every element b of B.

We have overall shown that $\mathscr{A}(F(B), A')$ is bijective to $\mathbf{Set}(B, U(A'))$, natural in both B and A'. This shows that the constructed functor F is left adjoint to the original functor U.

Exercise 6.3.23

(a)

Let P be a preordered set and let $\mathscr{P}$ be the corresponding category.

For every two elements x and y of P, let $x \sim y$ if and only if both $x \leq y$ and $y \leq x$. The relation $\sim$ is an equivalence relation on P.

- For every element x of P we have $x \leq x$, and therefore $x \sim x$. This shows that the relation $\sim$ is transitive.
- Let x and y be two elements of P. Both $x \sim y$ and $y \sim x$ are defined via the same two conditions, whence $x \sim y$ if and only if $y \sim x$. This shows that the relation $\sim$ is symmetric.
- Let x, y and z be three elements of P. Suppose that both $x \sim y$ and $y \sim z$. This means that

  $$x \leq y\,, \quad y \leq x\,, \quad y \leq z\,, \quad z \leq y\,.$$

  It follows from $x \leq y$ and $y \leq z$ that $x \leq z$, and it similarly follows from $z \leq y$ and $y \leq x$ that $z \leq x$. We have thus both $x \leq z$ and $z \leq x$, which shows that $x \sim z$. This shows that the relation $\sim$ is transitive.

Let x, x' and y, y' be elements of P with both $x \sim x'$ and $y \sim y'$. We claim that $x \leq y$ if and only if $x' \leq y'$. To prove this claim, it suffices to show that $x \leq y$ implies $x' \leq y'$, because $\sim$ is symmetric. So let's suppose that $x \leq y$. We know from the assumptions $x \sim x'$ and $y \sim y'$ that $x' \leq x$ and $y \leq y'$. Together with $x \leq y$, this tells us that indeed $x' \leq y'$.

Let Q be the quotient set $P/{\sim}$. We have thus seen that the preorder $\leq$ of P descends to a relation on Q, which we shall again denote by $\leq$. This induced relation satisfies

$$x \leq y \iff [x] \leq [y] \tag{6.13}$$

for any two elements x and y of P.

The relation $\leq$ on Q is both reflexive and transitive, since the original preorder on P is reflexive and transitive. In other words, $(Q, \leq)$ is again a preordered set. It is, in fact, already an ordered set. To see this, let $[x]$ and $[y]$ be two elements of Q with both $[x] \leq [y]$ and $[y] \leq [x]$. This means for the original two elements x and y of P that both $x \leq y$ and $y \leq x$. This tells us that $x \sim y$, and therefore that $[x] = [y]$.

Let $\mathscr{Q}$ be the category corresponding to the ordered set $(Q, \leq)$. The equivalence (6.13) tells us that the quotient map from P to Q extends to functor F from $\mathscr{P}$ to $\mathscr{Q}$ that is both full and faithful. The functor F is also surjective on objects, and thus overall an equivalence of categories. (Choosing an essential inverse to F corresponds to choosing a representative for each equivalence class of $\sim$.)

(b)

Let M be the class of morphisms of $\mathscr{A}$. We have for every set I the chain of isomorphisms and inclusions

$$M \supseteq \mathscr{A}(A, B^I) \cong \mathscr{A}(A, B)^I \supseteq \{f, g\}^I \cong \{0, 1\}^I \cong \mathscr{P}(I) \supseteq \{\{i\} \mid i \in I\} \cong I\,. \tag{6.14}$$

We have thus constructed for every set I an inclusion from I to M. (More explicitly, we assign to an element i of I the morphism from A to B^I whose i-th component is f, and whose other components are g.) This entails that M cannot be a set (since otherwise we could embed $\mathscr{P}(M)$ into M, which is not possible for cardinality reasons).

(c)

Let $\mathscr{A}$ be a category that is both small and complete. We find from part (b) of this exercise that for any two objects A and B of $\mathscr{A}$ there exists at most one morphism from A to B in $\mathscr{A}$. The category $\mathscr{A}$ therefore corresponds to a preordered set P. More explicitly, the elements of P are the objects of $\mathscr{A}$, and for any two such objects A and B, we have $A \leq B$ in P if and only if there exists a morphism from A to B in $\mathscr{A}$.

It follows from part (a) of this exercise that $\mathscr{A}$ is equivalent to an ordered set. This ordered set is complete since $\mathscr{A}$ is complete (and equivalence of categories preserves completeness).

(d)

Let $\mathscr{A}$ be a category that admits all finite products. Suppose that there exist two objects A and B of $\mathscr{A}$ for which there exist two distinct morphisms f and g from A to B. We then find from (6.14) that for every finite set I, the set of morphisms of $\mathscr{A}$ is of larger cardinality than I. This tells us that the category $\mathscr{A}$ contains infinitely many morphisms, and is therefore infinite.

Let now $\mathscr{B}$ be a category that is finite and admits all finite products. We have just seen that there exists for every two objects A and B of $\mathscr{B}$ at most one morphism from A to B in $\mathscr{B}$. In other words, the category $\mathscr{B}$ corresponds to a preordered set P. This preordered set P is finite, since $\mathscr{B}$ is finite. It follows from part (a) of this exercise (and our solution to it) that $\mathscr{B}$ is equivalent to an ordered set Q that is again finite. The ordered set Q admits finite products, because $\mathscr{B}$ admits finite products. Since Q is finite, this means that Q admits all small products. As seen somewhere throughout the book, this means that Q admits all small limits. In other words, Q is complete.

Exercise 6.3.24

(a)

We consider for every natural number n the set $W_n := (A \times \{1, -1\})^n$, and the evaluation map

$$e_n : W_n \longrightarrow G , \quad ((a_1, \varepsilon_1), \dots , (a_n, \varepsilon_n)) \longmapsto a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} .$$

For the set $W := \sum_{n \in \mathbb{N}} W_n$, the maps $e_n$assemble into a single map

$$e : W \longrightarrow G .$$

The subgroup H of G generated by A is precisely the image of the map e. We therefore find that

$$|H| \leq |W| \,.$$

We show in the following that $|W| \leq \max\{|\mathbb{N}|, |A|\}$. For this, we distinguish between two cases.

- Suppose that the set A is finite. It then follows that each set W_n is again finite. The set $W = \sum_{n \in \mathbb{N}} W_n$ is therefore countable (either finite or infinite), so that $|W| = |\mathbb{N}|$.
- Suppose that the set A is infinite. It then holds that

  $$|A \times \{1, -1\}| = |A| \cdot |\{1, -1\}| = |A| \cdot 2 = |A| + |A| = |A| \,,$$

  and it follows that $|W_n| = |A|^n = |A|$. It further follows that

  $$|W| = \sum_{n \in \mathbb{N}} |W_n| = \sum_{n \in \mathbb{N}} |A| = |A| \,.$$

(b)

For every set T let $G(T)$ be the set of group structures T. This collection is indeed a set, since it is a subset of $\mathbf{Set}(T \times T, T)$.

Let $\mathscr{G}$ be the class of groups who are of cardinality at most $|S|$. Every subset T of S and every element of $G(T)$ results in a group whose underlying set is T. We have in this way a map

$$\sum_{T \in \mathscr{P}(S)} G(T) \longrightarrow \mathscr{G} \,,$$

which further descends to a map

$$\sum_{T \in \mathscr{P}(S)} G(T) \longrightarrow \mathscr{G}/{\cong} \,.$$

Every group contained in $\mathscr{G}$ is isomorphic to a group whose underlying set is a subset of S, whence this second map is surjective. The left-hand side is a set, so it follows that the right-hand side is also one.

(c)

Let (G, g) be an object on $A \Rightarrow U$. This means that G is a group and g is a set-theoretic map from A to $U(G)$. Let H be the subgroup of G generated by the image of g, i.e., the subgroup of G generated by the family $(g(a))_{a \in A}$. The map g restricts to a set-theoretic map h from A to $U(H)$, resulting in the object (H, h) of $A \Rightarrow G$. The inclusion map i from H to G is a homomorphism of groups that makes the diagram

$$\begin{tikzcd} U(H) \arrow[rr, "U(i)"] & & U(G) \\ & A \arrow[ul, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

commute. In other words, i is a morphism from (H, h) to (G, g) in $A \Rightarrow U$.

Let S'' be the class of objects (H, h) of $A \Rightarrow G$ for which the group H is generated by the image of h. We have seen above that the class S'' is weakly initial in $A \Rightarrow U$.

We find from part (a) of this exercise that for every object (H, h) of S'', the cardinality of the group H is at most $\max\{|\mathbb{N}|, |A|\}$.

Let S be a set of cardinality $\max\{|\mathbb{N}|, |A|\}$, and let S' be the class of all those objects (G, g) of $A \Rightarrow U$ for which the group G has at most cardinality $|S|$. The class S' contains the class S'', and is therefore again weakly initial in $A \Rightarrow U$.

Let S be the set of all those objects (G, g) of $A \Rightarrow U$ for which the underlying set of the group G is a subset of S. Every object of the class S' is isomorphic to some object of the class S.[7] The set S is therefore again weakly initial in $A \Rightarrow U$.

(d)

The category **Grp** is locally small. We know from part (c) of this exercise that for every set A, the comma category $A \Rightarrow G$ admits a weakly initial set. We

[7] We use here a similar argument as in part (b). Instead, we could directly use part (b) as follows:

> By part (b), the collection of isomorphism classes of objects of S' is small. We can therefore choose S as a set of representatives for the isomorphism classes of objects of S'.

We haven't chosen this approach, so that we can avoid choosing representatives.

know that the category **Set** is complete, and we know from Exercise 5.3.11, part (a) that the functor U creates limits. It follows that **Grp** is complete and that U preserves limits.

It follows from the general adjoint functor theorem that the functor U admits a left adjoint.

Exercise 6.3.25

We abbreviate $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ as $\hat{\mathbf{A}}$, and denote the Yoneda embedding from $\mathbf{A}$ to $\hat{\mathbf{A}}$ by よ.

The Yoneda embedding preserves products

We know from Corollary 6.2.11 that the Yoneda embedding よ preserves limits, and therefore in particular products.

What it means to preserve exponentials

The book doesn't define what it means to preserve exponentials, so let us rectify this.

Suppose that we have two cartesian closed categories $\mathscr{A}$ and $\mathscr{B}$ and a functor F from $\mathscr{A}$ to $\mathscr{B}$. For F to preserve exponentials, we surely need that $F(C^B) \cong F(C)^{F(B)}$ for any two objects B and C of $\mathscr{A}$.

However, this won't be enough: constructions in category theory typically come with structure morphisms, and for a functor to preserve a certain kind of construction, we also want it to play nicely with these structure morphisms. (Recall, for example, what it means for F to preserve binary products: it is not only enough that $F(A \times B) \cong F(A) \times F(B)$ for any two objects A and B, even if these isomorphisms are natural in A and B.[8] More strictly than that, we require the functor F to turn the canonical projects $B \times C \to B$ and $B \times C \to C$ into the canonical projections $F(B) \times F(C) \to F(B)$ and $F(B) \times F(C) \to F(C)$.)

[8] To see this, let us consider the functor $F := (-) \times \mathbb{N}$ from **Set** to **Set**. This functor satisfies $F(B \times C) \cong F(B) \times F(C)$ for any two sets B and C because $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ as sets. By fixing one such bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we even get a natural isomorphism $F(-) \times F(-) \cong F((-) \times (-))$. But the functor F does not preserve products, since for any two non-empty sets B and C with canonical projections $p : B \times C \to B$ and $q : B \times C \to C$, the induced map $\langle p \times 1_{\mathbb{N}}, q \times 1_{\mathbb{N}} \rangle : B \times C \times \mathbb{N} \to (B \times \mathbb{N}) \times (C \times \mathbb{N})$ is not an isomorphism.

So to say what it means for F to preserve exponentials, we need to understand how the structure morphisms for exponentials look like.

In the book, the exponential C^B has been defined as the value of $(-)^B$ at C, where $(-)^B$ is a right-adjoint of $(-) \times B$. Part of the data of the adjoint $(-)^B$ is a natural isomorphism

$$\alpha : \mathscr{A}((-) \times B, -) \longrightarrow \mathscr{A}(-, (-)^B).$$

Part of the natural isomorphism α is a natural isomorphism

$$\alpha_{(-),C} : \mathscr{A}((-) \times B, C) \longrightarrow \mathscr{A}(-, C^B),$$

for every object C of $\mathscr{A}$, as explained in Exercise 1.3.29. The natural isomorphism $\alpha_{(-),C)}$ needs to be understood as being part of data of the exponential C^B.

In other words, the exponential C^B comes with a canonical natural isomorphism $\mathscr{A}((-) \times B, C) \Rightarrow \mathscr{A}(-, C^B)$. This natural isomorphism can also be characterized in terms of its universal element, as explained in Section 3.1. This universal element is an element ev of $\mathscr{A}(C^B \times B, C)$ such that for every object A of $\mathscr{A}$ and every element f of $\mathscr{A}(A \times B, C)$, there exists a unique morphism $f' : A \to C^B$ such that $(f' \times 1_B)^*(\mathrm{ev}) = f$. In other words: for every object A of $\mathscr{A}$ and every morphism of the form $f : A \times B \to C$ there exists a unique morphism of the form $f' : A \to C^B$ such that the following diagram commutes:

$$\begin{tikzcd} A \times B \arrow[d, "f' \times 1_B"'] \arrow[dr, "f"] & \\ C^B \times B \arrow[r, "\mathrm{ev}"'] & C \end{tikzcd}$$

We call this morphism ev the **canonical evaluation morphism**. This is the structure morphism belonging to the exponential C^B.

We can now say what it means for F to preserve exponentials. Suppose that F already preserves binary products. For every two objects B and C of $\mathscr{A}$ let C^B be the exponential from B to C with canonical evaluation morphism $\mathrm{ev}_{B,C} : C^B \times B \to C$. In $\mathscr{B}$, we have the induced morphism

$$\mathrm{ev}'_{F(B),F(C)} : F(C^B) \times F(B) \longrightarrow F(C^B \times B) \xrightarrow{F(\mathrm{ev}_{B,C})} F(C).$$

(The morphism $F(C^B) \times F(B) \to F(C^B \times B)$ is the isomorphism that comes from the fact that F preserves products.) We say that the functor F **preserves exponentials** if for any two objects B and C of $\mathscr{A}$ the object $F(C^B)$ together with the induced morphism $\mathrm{ev}'_{F(B),F(C)}$ is an exponential from $F(B)$ to $F(C)$.

Exponentials in presheaf categories

Let us revisit the construction of exponential objects in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ in a way that emphasizes the canonical evaluation morphisms. By a "presheaf" we will always mean a presheaf on $\mathbf{A}$.

Given two presheaves Y and Z, we once again define the presheaf Z^Y as

$$Z^Y := \hat{\mathbf{A}}(よ(-) \times Y, Z)\,.$$

For every morphism

$$f : B \longrightarrow A$$

in $\mathbf{A}$, the result of applying Z^Y to f is consequently the map

$$Z^Y(f) : \hat{\mathbf{A}}(よ(A) \times Y, Z) \longrightarrow \hat{\mathbf{A}}(よ(B) \times Y, Z)$$

given on elements by

$$Z^Y(f)(\theta) = (よ(f) \times 1_Y)^*(\theta) = \theta \circ (よ(f) \times 1_Y)$$

for every $\theta \in \hat{\mathbf{A}}(よ(A) \times Y, Z)$. The function value $Z^Y(f)(\theta)$ is a natural transformation from $よ(B) \times Y$ to Z, whose components are given by the maps

$$Z^Y(f)(\theta)_C : \mathbf{A}(C, B) \times Y(C) \longrightarrow Z(C)\,,$$

with

$$\begin{aligned} Z^Y(f)(\theta)_C(h, y) &= \big(\theta \circ (よ(f) \times 1_Y)\big)_C(h, y) \\ &= \big(\theta_C \circ (よ(f) \times 1_Y)_C\big)(h, y) \\ &= \big(\theta_C \circ (f_* \times 1_{Y(C)})\big)(h, y) \\ &= \theta_C\big((f_* \times 1_{Y(C)})(h, y)\big) \\ &= \theta_C(f \circ h, y) \end{aligned} \tag{6.15}$$

for every object C of $\mathbf{A}$ and every element (h, y) of $\mathbf{A}(C, B) \times Y(C)$.

To make the presheaf Z^Y into an exponential from Y to Z, we need to construct an evaluation natural transformation

$$\varepsilon : Z^Y \times Y \Longrightarrow Z\,.$$

For every object A of $\mathbf{A}$ we define the component ε_A as

$$\varepsilon_A : Z^Y(A) \times Y(A) \longrightarrow Z(A), \quad (\theta, y) \longmapsto \theta_A(1_A, y).$$

To check the naturality of the transformation ε, we consider an arbitrary morphism

$$f : B \longrightarrow A$$

in $\mathbf{A}$. The resulting diagram

$$\begin{tikzcd} Z^Y(A) \times Y(A) \arrow[r, "Z^Y(f)\times Y(f)"] \arrow[d, "\varepsilon_A"'] & Z^Y(B) \times Y(B) \arrow[d, "\varepsilon_B"] \\ Z(A) \arrow[r, "Z(f)"] & Z(B) \end{tikzcd}$$

commutes because for every element (θ, y) of the top-left corner, we have

$$\begin{aligned} \varepsilon_B\big((Z^Y(f) \times Y(f))(\theta, y)\big) &= \varepsilon_B\big(Z^Y(f)(\theta), Y(f)(y)\big) \\ &= Z^Y(f)(\theta)_B\big(1_B, Y(f)(y)\big) \\ &= \theta_B\big(f \circ 1_B, Y(f)(y)\big) \\ &= \theta_B(f, Y(f)(y)), \end{aligned}$$

as well as

$$\begin{aligned} Z(f)(\varepsilon_A(\theta, y)) &= Z(f)(\theta_A(1_A, y)) \\ &= \theta_B\big((よ(A) \times Y)(f)(1_A, y)\big) \\ &= \theta_B\big((よ(A)(f) \times Y(f))(1_A, y)\big) \\ &= \theta_B\big((f^* \times Y(f))(1_A, y)\big) \\ &= \theta_B(f, Y(f)(y)) \end{aligned}$$

because θ is a natural transformation from $よ(A) \times Y$ to Z.

We now need to show that for every presheaf X on $\mathbf{A}$ and every natural transformation

$$\beta : X \times Y \Longrightarrow Z$$

there exists a unique natural transformation

$$\alpha : X \Longrightarrow Z^Y$$

that makes the diagram

$$\begin{tikzcd} X \times Y \arrow[d, Rightarrow, "\alpha \times 1_Y"'] \arrow[dr, Rightarrow, "\beta"] & \\ Z^Y \times Y \arrow[r, Rightarrow, "\varepsilon"'] & Z \end{tikzcd} \tag{6.16}$$

commute.

Let us start by showing the uniqueness of α. The commutativity of the diagram (6.16) means that for every object A of $\mathbf{A}$ the diagram

$$\begin{tikzcd} X(A) \times Y(A) \arrow[d, "\alpha_A \times 1_{Y(A)}"'] \arrow[dr, "\beta_A"] & \\ Z^Y(A) \times Y(A) \arrow[r, "\varepsilon_A"'] & Z(A) \end{tikzcd}$$

commutes. In terms of elements, this is equivalent to saying that

$$\beta_A(x, y) = \varepsilon_A((\alpha_A \times 1_{Y(A)})(x, y)) = \varepsilon_A(\alpha_A(x), y) = \alpha_A(x)_A(1_A, y) \tag{6.17}$$

for every object A of $\mathbf{A}$ and all elements $x \in X(A)$, $y \in Y(A)$.

We want to determine the natural transformation $\alpha : X \Rightarrow Z^Y$ in question, which means that we need to determine for every object A of $\mathbf{A}$ its component

$$\alpha_A : X(A) \longrightarrow Z^Y(A) = \hat{\mathbf{A}}(よ(A) \times Y, Z) .$$

This means that we need to determine for every element x of $X(A)$ the element $\alpha_A(x)$ of $\hat{\mathbf{A}}(よ(A) \times Y, Z)$. This element is itself again a natural transformation, now from $よ(A) \times Y$ to Z. We hence need to determine for every object B of $\mathbf{A}$ the component

$$\alpha_A(x)_B : \mathbf{A}(B, A) \times Y(B) \longrightarrow Z(B) .$$

For this, we need to determine the value $\alpha_A(x)_B(f, y)$ for every element (f, y) of $\mathbf{A}(B, A) \times Y(B)$. The element f of $\mathbf{A}(B, A)$ is a morphism from B to A in $\mathbf{A}$, and therefore gives us the following commutative diagram:

$$\begin{tikzcd} X(A) \arrow[r, "X(f)"] \arrow[d, "\alpha_A"'] & X(B) \arrow[d, "\alpha_B"] \\ Z^Y(A) \arrow[r, "Z^Y(f)"] & Z^Y(B) \end{tikzcd}$$

We find from the commutativity of this diagram that

$$
\begin{aligned}
\alpha_A(x)_B(f, y) &= \alpha_A(x)_B(f \circ 1_B, y) \\
&= Z^Y(f)(\alpha_A(x))_B(1_B, y) \qquad (6.18) \\
&= \alpha_B(X(f)(x))_B(1_B, y) \qquad (6.19) \\
&= \beta_B(X(f)(x), y)\,,
\end{aligned}
$$

where equality (6.18) follows from the description of $Z^Y(f)$ from (6.15), and equality (6.19) follows from the property of α from (6.17). We have thus shown the uniqueness of α.

Let us now show the existence of the natural transformation α. We set

$$\alpha_A(x)_B(f, y) \coloneqq \beta_B(X(f)(x), y) \qquad (6.20)$$

for every two objects A and B of $\mathbf{A}$ and all $x \in X(A)$, $(f, y) \in \mathbf{A}(B, A) \times Y(B)$. This gives us a well-defined map

$$\alpha_A(x)_B : \mathbf{A}(B, A) \times Y(B) \longrightarrow Z(B)$$

for every two objects A and B of $\mathbf{A}$ and every $x \in X(A)$.

Let us show that the resulting transformation $\alpha_A(x)$ from $よ(A) \times Y$ to Z is natural for every object A of $\mathbf{A}$ and every $x \in X(A)$. For this, we need to show that for every morphism $g : C \to B$ in $\mathbf{A}$ the diagram

$$
\begin{tikzcd}
\mathbf{A}(B, A) \times Y(B) \arrow[r, "g^* \times Y(g)"] \arrow[d, "\alpha_A(x)_B"'] & \mathbf{A}(C, A) \times Y(C) \arrow[d, "\alpha_A(x)_C"] \\
Z(B) \arrow[r, "Z(g)"] & Z(C)
\end{tikzcd}
$$

commutes. This holds because

$$
\begin{aligned}
\alpha_A(x)_C\big((g^* \times Y(g))(f, y)\big) &= \alpha_A(x)_C\big(f \circ g, Y(g)(y)\big) \\
&= \beta_C\big(X(f \circ g)(x), Y(g)(y)\big) \\
&= \beta_C\big(X(g)(X(f)(x)), Y(g)(y)\big) \\
&= \beta_C\big((X(g) \times Y(g))(X(f)(x), y)\big) \\
&= \beta_C\big((X \times Y)(g)(X(f)(x), y)\big) \\
&= Z(g)\big(\beta_B(X(f)(x), y)\big) \\
&= Z(g)\big(\alpha_A(x)_B(f, y)\big)
\end{aligned}
$$

for every element (f, y) of the top-left corner of this diagram.

We have thus constructed a well-defined map

$$\alpha_A : X(A) \longrightarrow \hat{\mathbf{A}}(よ(A) \times Y, Z) = Z^Y(A)$$

for every object A of $\mathbf{A}$. Let us show that the resulting transformation α from X to Z^Y is natural. For this, we need to check that for every morphism $g : B \to A$ in $\mathbf{A}$ the following diagram commutes:

$$\begin{tikzcd}
X(A) \arrow[r, "X(g)"] \arrow[d, "\alpha_A"'] & X(B) \arrow[d, "\alpha_B"] \\
Z^Y(A) \arrow[r, "Z^Y(g)"] & Z^Y(B)
\end{tikzcd}$$

We hence need to show that

$$\alpha_B(X(g)(x))_C(f, y) = Z^Y(g)(\alpha_A(x))_C(f, y)$$

for every $x \in X(A)$, every object C of $\mathbf{A}$ and every $(f, y) \in \mathbf{A}(C, B) \times Y(C)$. This equality holds because

$$\begin{aligned}
\alpha_B(X(g)(x))_C(f, y) &= \beta_C\big(X(f)(X(g)(x)), y\big) \\
&= \beta_C(X(g \circ f)(x), y) \\
&= \alpha_A(x)_C(g \circ f, y) \\
&= Z^Y(g)(\alpha_A(x))_C(f, y) .
\end{aligned}$$

We have thus shown that α is natural.

It remains to show that the constructed natural transformation α makes the diagram (6.16) commute. We have already seen in (6.17) that the commutativity of (6.16) is equivalent to the equations

$$\beta_A(x, y) = \alpha_A(x)_A(1_A, y)$$

for every object A of $\mathbf{A}$ and all $x \in X(A)$, $y \in Y(A)$. The natural transformation α satisfies these equations because

$$\alpha_A(x)_A(1_A, y) = \beta_A(X(1_A)(x), y) = \beta_A(1_{X(A)}(x), y) = \beta_A(x, y) .$$

The Yoneda embedding preserves exponentials

Let us now show that the Yoneda embedding preserves exponentials. Let B and C be two objects of $\mathbf{A}$ with exponential C^B and evaluation homomorphism

$$\mathrm{ev} : \; C^B \times B \longrightarrow C\,.$$

We need to show that the object $\text{よ}(C^B)$ together with the induced natural transformation

$$\mathrm{ev}' : \; \text{よ}(C^B) \times \text{よ}(B) \longrightarrow \text{よ}(C^B \times B) \xrightarrow{\mathrm{ev}_*} \text{よ}(C)$$

as an exponential from $\text{よ}(B)$ to $\text{よ}(C)$. We note that the components of ev' are given by

$$\mathrm{ev}'_D : \; \mathbf{A}(D, C^B) \times \mathbf{A}(D, B) \longrightarrow \mathbf{A}(D, C)\,, \quad (f, g) \longmapsto \mathrm{ev} \circ \langle f, g \rangle$$

for every object D of $\mathbf{A}$.

We consider the exponential $\text{よ}(C)^{\text{よ}(B)}$ and its evaluation natural transformation

$$\varepsilon : \; \text{よ}(C)^{\text{よ}(B)} \times \text{よ}(B) \Longrightarrow \text{よ}(C)\,.$$

There exists a unique natural transformation

$$\alpha : \; \text{よ}(C^B) \Longrightarrow \text{よ}(B)^{\text{よ}(C)}$$

that makes the diagram

$$\begin{tikzcd}
\text{よ}(C^B) \times \text{よ}(C) \arrow[d, Rightarrow, "\alpha \times 1_{\text{よ}(C)}"'] \arrow[dr, Rightarrow, "\mathrm{ev}'"] & \\
\text{よ}(C)^{\text{よ}(B)} \times \text{よ}(C) \arrow[r, Rightarrow, "\varepsilon"'] & \text{よ}(B)
\end{tikzcd}$$

commute. We show in the following that this natural transformation α is already an isomorphism. Since $\text{よ}(C)^{\text{よ}(B)}$ together with ε is an exponential from $\text{よ}(B)$ to $\text{よ}(C)$, this then shows that $\text{よ}(C^B)$ together with ev' is also an exponential from $\text{よ}(B)$ to $\text{よ}(C)$. To show that α is an isomorphism, we will show that it is an isomorphism in each component.

We can already see that the presheaves $よ(C)^{よ(B)}$ and $よ(C^B)$ are isomorphic in each component (in some way) because

$$
\begin{aligned}
よ(C^B)(A) &= \mathbf{A}(A, C^B) \\
&\cong \mathbf{A}(A \times B, C) \\
&\cong \hat{\mathbf{A}}(よ(A \times B), よ(C)) \\
&\cong \hat{\mathbf{A}}(よ(A) \times よ(B), よ(C)) \\
&= よ(C)^{よ(B)}(A)
\end{aligned}
$$

for every object A of $\mathbf{A}$. Let us denote this isomorphism from $よ(C^B)(A)$ to $よ(C)^{よ(B)}(A)$ by φ_A. We can derive an explicit formula for φ_A as follows:

- Let f be an element of $よ(C^B)(A) = \mathscr{A}(A, C^B)$.
- The resulting element f' of $\mathbf{A}(A \times B, C)$ is given by $f' = \mathrm{ev} \circ (f \times 1_B)$.
- The resulting element γ of $\hat{\mathbf{A}}(よ(A \times B), よ(C))$ is a natural transformation from $よ(A \times B)$ to $よ(C)$, which is given in components by

  $$\gamma_D : \mathbf{A}(D, A \times B) \longrightarrow \mathbf{A}(D, C), \quad g \longmapsto f' \circ g$$

  for every object D of $\mathbf{A}$. In other words,

  $$\gamma_D(g) = \mathrm{ev} \circ (f \times 1_B) \circ g .$$

- The resulting element γ' of $\hat{\mathbf{A}}(よ(A) \times よ(B), よ(C) = よ(C)^{よ(B)}(A)$ is a natural transformation from $よ(A) \times よ(B)$ to $よ(C)$, which is given in components by

  $$\gamma'_D : \mathbf{A}(D, A) \times \mathbf{A}(D, B) \longrightarrow \mathbf{A}(D, C), \quad (g, h) \longmapsto \gamma_D(\langle g, h \rangle)$$

  for every object D of $\mathbf{A}$. In other words,

  $$\gamma'_D(g, h) = \mathrm{ev} \circ (f \times 1_B) \circ \langle g, h \rangle = \mathrm{ev} \circ \langle f \circ g, h \rangle .$$

We see altogether that the isomorphism φ_A is given by

$$\varphi_A(f)_D(g, h) = \mathrm{ev} \circ \langle f \circ g, h \rangle$$

for every object D of $\mathbf{A}$.

We have previously seen an explicit formula for α in (6.20). Adapting this general formula to our special case, we see that for every object A of $\mathbf{A}$, the component α_A is given by

$$\alpha_A(f)_D(g,h) = \mathrm{ev}'_D(よ(C^B)(g)(f), h) = \mathrm{ev}'_D(f \circ g, h) = \mathrm{ev} \circ \langle f \circ g, h \rangle\,.$$

We hence see that the component α_A is precisely the isomorphism φ_A, which entails that α_A is an isomorphism.

Exercise 6.3.26

The construction of subobjects from Exercise 5.1.40 allows us to generalize the notion of "subsets" to arbitrary categories. One important operation that we can do with subsets is taking preimages: given two sets A' and A and a map f from A' to A, we can form for every subset S of A its preimage $f^{-1}(S)$, which is a subset of A'. The following proposition allows us to express preimages in the language of category theory.

Proposition 6.G. Let A be set, let S be a subset of A and let i be the inclusion map from S to A. Let A' be another set and let f a map from A' to A. Let i' the inclusion map from $f^{-1}(S)$ to A' and let f' the restriction of f to a map from $f^{-1}(S)$ to S. The following diagram is a pullback diagram:

$$\begin{tikzcd}
f^{-1}(S) \arrow[r, "f'"] \arrow[d, "i'"'] & S \arrow[d, "i"] \\
A' \arrow[r, "f"] & A
\end{tikzcd}$$

In the following, we will use this reformulation of preimages via pullbacks to generalize preimages to arbitrary categories.

(a)

We can form for every object (X, m) of $\mathbf{Monic}(A)$ the pullback diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "m'"'] & X \arrow[d, "m"] \\
A' \arrow[r, "f"] & A
\end{tikzcd} \tag{6.21}$$

We know from Exercise 5.1.42 that the morphism m' is again a monomorphism, whence we have constructed an object (X', m') of $\mathbf{Monic}(A')$. By choosing for every object (X, m) of $\mathbf{Monic}(A)$ a pullback as above, we have thus constructed a map

$$\mathbf{Monic}(f) : \ \mathrm{Ob}(\mathbf{Monic}(A)) \longrightarrow \mathrm{Ob}(\mathbf{Monic}(A')) .$$

We show in the following that the map $\mathbf{Monic}(f)$ descends to a map

$$\mathrm{Sub}(f) : \ \mathrm{Sub}(A) \longrightarrow \mathrm{Sub}(A') .$$

This then shown that we can pull back subobjects of A to subobjects of A' via pullbacks.

To show that the map $\mathbf{Monic}(f)$ descends to a map $\mathrm{Sub}(f)$ as desired, we need to show that isomorphic objects of $\mathbf{Monic}(A)$ lead to isomorphic objects in $\mathbf{Monic}(A')$. We will show more conceptually that $\mathbf{Monic}(f)$ extends to a functor from $\mathbf{Monic}(A)$ to $\mathbf{Monic}(A')$.

To construct the action of $\mathbf{Monic}$ on morphisms, let

$$h : \ (X, m) \longrightarrow (Y, n)$$

be a morphism in $\mathbf{Monic}(A)$. This means that we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "m"'] & & Y \arrow[dl, "n"] \\
& A &
\end{tikzcd}$$

Let (X', m') and (Y', n') be the images of (X, m) and (Y, n) under $\mathbf{Monic}(f)$. There exists a unique morphism from X' to Y' that makes the following diagram commute:

$$\begin{tikzcd}
X' \arrow[rr] \arrow[dr, dashed] \arrow[dd, "m'"'] & & X \arrow[dr, "g"] \arrow[dd, "m" near start] & \\
& Y' \arrow[rr] \arrow[dl, "n'"] & & Y \arrow[dl, "n"] \\
A' \arrow[rr, "f"] & & A &
\end{tikzcd}$$

We denote this induced morphism from X' to Y' by $\mathbf{Monic}(f)(h)$, which is a morphism from (X', m') to (Y', n'). This morphism is thus unique with making the following two-dimensional diagram commute:

$$\begin{tikzcd}
A' & X' \arrow[l, "m'"'] \arrow[r] \arrow[d, "\mathbf{Monic}(f)(h)"] & X \arrow[d, "h"] \\
A' \arrow[u, equal] & Y' \arrow[l, "n'"'] \arrow[r] & Y
\end{tikzcd}$$

Let us show that the assignment $\mathbf{Monic}(f)$ from $\mathbf{Monic}(A)$ to $\mathbf{Monic}(A')$ is functorial.

- Let (X, m) be an object of $\mathbf{Monic}(A)$, and let (X', m') be resulting object of $\mathbf{Monic}(A')$. The diagram

  $$\begin{tikzcd}
  A' & X' \arrow[l, "m'"'] \arrow[r] \arrow[d, "1_{X'}"] & X \arrow[d, "1_X"] \\
  A' \arrow[u, equal] & X' \arrow[l, "n'"'] \arrow[r] & X
  \end{tikzcd}$$

  commutes, whence the morphism $1_{X'} = 1_{(X',m')}$ satisfies the defining relationship of the morphism $\mathbf{Monic}(f)(1_{(X,n)})$.
- Let (X, m), (Y, n) and (Z, p) be objects of $\mathbf{Monic}(A)$, and let (X', m'), (Y', n') and (Z', p') be the resulting objects of $\mathbf{Monic}(A')$. Let

  $$h : (X, m) \longrightarrow (Y, n), \quad k : (Y, n) \longrightarrow (Z, p)$$

  be two composable morphisms in $\mathbf{Monic}(A)$. We have the following commutative diagram:

  $$\begin{tikzcd}
  A' & X' \arrow[l, "m'"'] \arrow[r] \arrow[d, "\mathbf{Monic}(f)(h)"] & X \arrow[d, "h"] \\
  A' \arrow[u, equal] & Y' \arrow[l, "n'"'] \arrow[r] \arrow[d, "\mathbf{Monic}(f)(k)"] & Y \arrow[d, "k"] \\
  A' \arrow[u, equal] & Z' \arrow[l, "p'"'] \arrow[r] & Z
  \end{tikzcd}$$

By leaving out the middle row of this diagram, we end up with the following commutative diagram:

$$
\begin{tikzcd}
A' \arrow[d, equal] & X' \arrow[l, "m'"'] \arrow[r] \arrow[d, "\mathbf{Monic}(f)(k)\circ\mathbf{Monic}(f)(h)"] & X \arrow[d, "k\circ h"] \\
A' & Z' \arrow[l, "p'"'] \arrow[r] & Z
\end{tikzcd}
$$

The composite $\mathbf{Monic}(f)(k) \circ \mathbf{Monic}(f)(h)$ thus satisfies the defining property of the morphism $\mathbf{Monic}(f)(k \circ h)$. These morphisms are therefore the same.

We have altogether constructed a functor

$$\mathbf{Monic}(f) : \ \mathbf{Monic}(A) \longrightarrow \mathbf{Monic}(A') \,.$$

This functor induces a map

$$\mathrm{Sub}(f) : \ \mathrm{Sub}(A) \longrightarrow \mathrm{Sub}(A')$$

as desired. This map $\mathrm{Sub}(f)$ has the following explicit description: Given an element $[(X,m)]$ of $\mathrm{Sub}(A)$, we form the pullback diagram (6.21). The image of $[(X,m)]$ under $\mathrm{Sub}(f)$ is then the element $[(X',m')]$.

Remark 6.H. The functor $\mathbf{Monic}(f)$ that we constructed above is somewhat evil, since it relies on choosing pullbacks. However, pullbacks are unique up to isomorphism. We can use this uniqueness up to isomorphism to show that the functor $\mathbf{Monic}(f)$ is again unique up to isomorphism. That is, different choices of pullbacks result in isomorphic functors.

As a consequence, the induced functor $\mathrm{Sub}(f)$ from $\mathrm{Sub}(A)$ to $\mathrm{Sub}(A')$ does not depend on our original choice of pullbacks.

(b)

The collection $\mathrm{Sub}(A)$ is a set for every object A of $\mathscr{A}$, since the category $\mathscr{A}$ is assumed to be well-powered. We can therefore regard the construction Sub as an assignment from $\mathscr{A}$ to **Set** (both on objects and on morphisms). Let us check that this assignment is contravariantly functorial.

- Let A be an object of $\mathscr{A}$ and consider the identity morphism

$$1_A : A \longrightarrow A .$$

For every object (X, m) of $\mathbf{Monic}(A)$, the diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "m"'] & X \arrow[d, "m"] \\ A \arrow[r, "f"] & A \end{tikzcd}$$

is a pullback diagram. Therefore,

$$\mathrm{Sub}(1_A)([(X, m)]) = [(X, m)] .$$

This shows that $\mathrm{Sub}(1_A) = 1_{\mathrm{Sub}(A)}$.

- Let

$$f : A' \longrightarrow A , \quad f' : A'' \longrightarrow A'$$

be two composable morphisms in $\mathscr{A}$. Let (X, m) be an object of $\mathbf{Monic}(A)$, let (X', m') be the image of (X, m) under $\mathbf{Monic}(f)$, and let (X'', m'') be the image of (X', m') under $\mathbf{Monic}(f')$. This means that in the following commutative diagram, both squares are are pullback diagrams:

$$\begin{tikzcd} X'' \arrow[r] \arrow[d, "m''"'] & X' \arrow[r] \arrow[d, "m'"] & X \arrow[d, "m"] \\ A'' \arrow[r, "f'"] & A' \arrow[r, "f"] & A \end{tikzcd}$$

By leaving out the middle column of this diagram, we get the following commutative diagram:

$$\begin{tikzcd} X'' \arrow[r] \arrow[d, "m''"'] & X \arrow[d, "m"] \\ A'' \arrow[r, "f \circ f'"] & A \end{tikzcd}$$

It follows from Exercise 5.1.35 that this diagram is again a pullback diagram. We hence find that

$$\mathrm{Sub}(f \circ f')([(X, m)])) = [(X'', m'')] .$$

But we also have

$$[(X'', m'')] = \mathrm{Sub}(f')([(X', m')]) = \mathrm{Sub}(f')(\mathrm{Sub}(f)([(X, m)]))\,.$$

We have therefore shown that $\mathrm{Sub}(f \circ f') = \mathrm{Sub}(f') \circ \mathrm{Sub}(f)$.

We have shown that Sub is a contravariant functor from $\mathscr{A}$ to **Set**.

Remark 6.I. We should actually be able to lift the identities

$$\mathrm{Sub}(1_A) = 1_{\mathrm{Sub}(A)}\,, \quad \mathrm{Sub}(f \circ f') = \mathrm{Sub}(f') \circ \mathrm{Sub}(A)$$

to isomorphisms

$$\mathbf{Monic}(1_A) \cong 1_{\mathbf{Monic}(A)}\,, \quad \mathbf{Monic}(f \circ f') \cong \mathbf{Monic}(f') \circ \mathbf{Monic}(f)\,.$$

But we cannot expect these isomorphisms to be equalities, since these action of **Monic** on morphisms is only unique up to isomorphism.

(c)

Subobjects in **Set** are the same as subsets. More precisely, we have for every set A a bijection

$$\alpha_A : \mathscr{P}(A) \longrightarrow \mathrm{Sub}(A)\,, \quad S \longmapsto [(S, i_S)]\,,$$

where for every subset S of A we denote the inclusion map from S to A as i_S. The existence of these bijections also shows that the category **Set** is well-powered.

Let us check that the bijection α_A is natural in A, so that α is a natural isomorphism from $\mathscr{P}$ to Sub. We need to check that for every map

$$f : A' \longrightarrow A$$

the following diagram commutes:

$$\begin{tikzcd}
\mathscr{P}(A) \arrow[r, "\alpha_A"] \arrow[d, "\mathscr{P}(f)"'] & \mathrm{Sub}(A) \arrow[d, "\mathrm{Sub}(f)"] \\
\mathscr{P}(A') \arrow[r, "\alpha_{A'}"] & \mathrm{Sub}(A')
\end{tikzcd}$$

In view of our explicit description of $\operatorname{Sub}(f)$ via pullbacks, this commutativity follows from Proposition 6.G (page 224).

Let now Ω be the set $\{0, 1\}$. Given a subset S of a set A, we can consider the characteristic function of S on A. This is the map

$$\chi_{A,S} : A \longrightarrow \Omega\,, \qquad a \longmapsto \begin{cases} 1 & \text{if } a \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting map

$$\chi_A : \mathscr{P}(A) \longrightarrow \mathbf{Set}(A, \Omega)\,, \qquad S \longmapsto \chi_{A,S}$$

is bijective for every set A. It is also natural in A. To see this, we need to convince ourselves that for every map

$$f : A' \longrightarrow A$$

between sets, the following diagram commutes:

$$\begin{tikzcd} \mathscr{P}(A) \arrow[r, "\chi_A"] \arrow[d, "\mathscr{P}(f)"'] & \mathbf{Set}(A, \Omega) \arrow[d, "f^*"] \\ \mathscr{P}(A') \arrow[r, "\chi_{A'}"] & \mathbf{Set}(A', \Omega) \end{tikzcd}$$

This diagram commutes because we have for every subset S of A and every element a' of A' the equalities

$$f^*(\chi_A(S))(a') = f^*(\chi_{A,S})(a') = (\chi_{A,S} \circ f)(a') = \chi_{A,S}(f(a'))$$

and

$$\chi_{A', f^{-1}(S)}(a') = \chi_{A'}(f^{-1}(S))(a') = \chi_{A'}(\mathscr{P}(f)(S))(a')\,,$$

therefore the chain of equivalences

$$\begin{aligned} f^*(\chi_A(S))(a') = 1 &\iff \chi_{A,S}(f(a')) = 1 \\ &\iff f(a') \in S \\ &\iff a' \in f^{-1}(S) \\ &\iff \chi_{A', f^{-1}(S)}(a') = 1 \\ &\iff \chi_{A'}(\mathscr{P}(f)(S))(a') = 1\,, \end{aligned}$$

and thus the equality $f^*(\chi_A(S)) = \chi_{A'}(\mathscr{P}(f)(S))$.

We have overall constructed isomorphisms

$$\mathrm{Sub} \cong \mathscr{P} \cong \mathbf{Set}(-, \Omega)\,.$$

The existence of the composite isomorphism $\mathrm{Sub} \cong \mathbf{Set}(-, \Omega)$ shows that the functor Sub is represented by the set $\Omega = \{0, 1\}$.

Exercise 6.3.27

We denote the presheaf category $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ by $\hat{\mathbf{A}}$.

To give an explicit description of subobjects in $\hat{\mathbf{A}}$ we introduce the notion of a subfunctor: a subfunctor is to a functor what a subset is to a set. Just as subobjects in **Set** correspond to subsets, we will show that subobjects in $\hat{\mathbf{A}}$ correspond to subfunctors.

Definition of subfunctors

Let $\mathscr{A}$ be a category and let H be a functor from $\mathscr{A}$ to **Set**. We say that another functor S from $\mathscr{A}$ to **Set** is a **subfunctor** of H if it satisfies the following two conditions.

1. $S(A)$ is a subset of $H(A)$ for every object A of $\mathscr{A}$.
2. $S(f)$ is the restriction of $H(f)$ for every morphism f in $\mathscr{A}$.

In other words, a subfunctor S of H consists of a subset $S(A)$ of $H(A)$ for every object A of $\mathscr{A}$, such that $H(f)(S(A)) \subseteq S(B)$ for every morphism $f : A \to B$ in $\mathscr{A}$.

We write $S \subseteq H$ if S is a subfunctor of H. The relation $\subseteq$ is a partial order on the class of functors from $\mathscr{A}$ to **Set**. We denote the class of subfunctors of H by $\mathscr{P}(H)$. Given two subfunctors S and T of H, we have $S \subseteq T$ if and only if $S(A) \subseteq T(A)$ for every object A of $\mathscr{A}$.

The image of a natural transformation

Given another functor H' from $\mathscr{A}$ to **Set** and a natural transformation α from H' to H, we define a subfunctor $\mathrm{im}(\alpha)$ of H with

$$\mathrm{im}(\alpha)(A) \coloneqq \mathrm{im}(\alpha_A)$$

for every object A of $\mathscr{A}$. This is indeed a subfunctor of H because

$$\begin{aligned} H(f)\big(\mathrm{im}(\alpha)(A)\big) &= H(f)(\mathrm{im}(\alpha_A)) \\ &= H(f)(\alpha_A(F(A))) \\ &= \alpha_B\big(F(f)(F(A))\big) \\ &\subseteq \alpha_B(F(B)) \\ &= \mathrm{im}(\alpha_B) \\ &= \mathrm{im}(\alpha)(B) \end{aligned}$$

for every morphism $f\colon A \to B$ in $\mathscr{A}$. We refer to the subfunctor $\mathrm{im}(\alpha)$ of H as the **image** of α.

Correspondence between subobjects and subfunctors

Let us now restrict our attention to subobjects of H. We claim that the map

$$\mathrm{Ob}(\mathbf{Monic}(H)) \longrightarrow \mathscr{P}(H)\,, \quad (X, \mu) \longmapsto \mathrm{im}(\mu)\,, \tag{6.22}$$

descends to a well-defined bijection from $\mathrm{Sub}(H)$ to $\mathscr{P}(H)$. To prove this claim we need to check the following subclaims:

1. Isomorphic objects of $\mathbf{Monic}(H)$ have the same image.
2. The map (6.22) is surjective.
3. If two objects of $\mathbf{Monic}(H)$ have the same image, then they are already isomorphic in $\mathbf{Monic}(A)$.

To show the first subclaim, suppose that we have a commutative diagram of functors and natural transformations as follows:

$$\begin{tikzcd} & H & \\ F \arrow[ur, Rightarrow, "\alpha"] \arrow[rr, Rightarrow] & & G \arrow[ul, Rightarrow, "\beta"'] \end{tikzcd}$$

We then have $\mathrm{im}(\alpha) \subseteq \mathrm{im}(\beta)$. For two isomorphic objects (X, μ) and (X, μ') of $\mathbf{Monic}(H)$ we thus have both $\mathrm{im}(\mu) \subseteq \mathrm{im}(\mu')$ and $\mathrm{im}(\mu') \subseteq \mathrm{im}(\mu)$, and therefore $\mathrm{im}(\mu) = \mathrm{im}(\mu')$.

Now we show the second subclaim. For every subfunctor S of H we have a natural transformation $\iota\colon S \Rightarrow H$ whose component ι_A is given by the inclusion map from $S(A)$ to $H(A)$ for every object A of $\mathscr{A}$. Each component of ι is an injective map, whence ι is a monomorphism. The object (S, ι) of $\mathbf{Monic}(H)$ serves as a preimage of S for the map (6.22), whence this map is surjective.

To show the third subclaim, suppose that (X, μ) and (X', μ') are two objects of $\mathbf{Monic}(H)$ with the same image in $\mathscr{P}(H)$, i.e., with $\operatorname{im}(\mu) = \operatorname{im}(\mu')$. We know from Exercise 6.2.20 that the components of μ and μ' are injective. For every object A of $\mathscr{A}$ we have thefore the two injective maps

$$\mu_A\colon X(A) \longrightarrow H(A), \quad \mu'_A\colon X'(A) \longrightarrow H(A).$$

These two maps have the same image since

$$\operatorname{im}(\mu_A) = \operatorname{im}(\mu)(A) = \operatorname{im}(\mu')(A) = \operatorname{im}(\mu'_A).$$

As seen in the solution to Exercise 5.1.40, there exists a unique bijection α_A from $X(A)$ to $X'(A)$ that makes the following diagram commute:

$$\begin{tikzcd} X(A) \arrow[rr, "\alpha_A"] \arrow[dr, "\mu_A"'] & & X'(A) \arrow[dl, "\mu'_A"] \\ & H(A) & \end{tikzcd}$$

We claim that the resulting transformation α from X to X' is natural. To show this, we need to check that for every morphism $f\colon A \to B$ in $\mathscr{A}$ the diagram

$$\begin{tikzcd} X(A) \arrow[r, "\alpha_A"] \arrow[d, "X(f)"'] & X'(A) \arrow[d, "X'(f)"] \\ X(B) \arrow[r, "\alpha_B"] & X'(B) \end{tikzcd}$$

commutes. To this end, we consider the following extended diagram:

$$
\begin{tikzcd}
X(A) \arrow[rr, "\alpha_A"] \arrow[dd, "X(f)"'] \arrow[dr, "\mu_A"'] & & X'(A) \arrow[dd, "X'(f)"] \arrow[dl, "\mu'_A"] \\
& H(A) \arrow[dd, "H(f)"] & \\
X(B) \arrow[rr, "\alpha_B"] \arrow[dr, "\mu_B"'] & & X'(B) \arrow[dl, "\mu'_B"] \\
& H(B) &
\end{tikzcd}
$$

The top and bottom of this diagram commute by definition of α, and the frontal two sides commute by the naturality of μ and μ'. It follows that

$$
\begin{aligned}
\mu'_B \circ X'(f) \circ \alpha_A &= H(f) \circ \mu'_A \circ \alpha_A \\
&= H(f) \circ \mu_A \\
&= \mu_B \circ X(f) \\
&= \mu'_B \circ \alpha_B \circ X(f) \,.
\end{aligned}
$$

The map μ'_B is injective, so it follows that $X'(f) \circ \alpha_A = \alpha_B \circ X(f)$, as desired.

We have thus constructed a natural isomorphism α from X to X' that makes the diagram

$$
\begin{tikzcd}
X \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\mu"'] & & X' \arrow[dl, Rightarrow, "\mu'"] \\
& H &
\end{tikzcd}
$$

commute. It follows from Proposition 5.B (page 131) that this natural isomorphism α is an isomorphism in $\mathbf{Mon}(H)$ from (X, μ) to (X', μ'). This shows the third subclaim.

We have thus constructed a bijection from $\mathrm{Sub}(H)$ to $\mathscr{P}(H)$. In this way, subobjects of H are the same as subfunctors of H.

The category $\hat{\mathrm{A}}$ is well-powered

We have for every functor H from $\mathscr{A}$ to **Set** an injective map from $\mathscr{P}(H)$ into the product $\prod_{A \in \mathrm{Ob}(\mathscr{A})} H(A)$. So if the category $\mathscr{A}$ is small (or more generally if $\mathrm{Ob}(\mathscr{A})$ is a set), then it follows that $\mathscr{P}(H)$ is a set, whence $\mathrm{Sub}(H)$ is a set. This tells us that the category $\hat{\mathrm{A}}$ is well-powered.

Functoriality of the power set

Let H and H' be two functors from $\mathscr{A}$ to **Set**. Suppose we have a subfunctor S' of H' and a natural transformation α from H to H'. We can then consider for every object A of $\mathscr{A}$ the subset $S(A) \coloneqq \alpha^{-1}(S'(A))$ of $H(A)$. These sets form a subfunctor S of H since for every morphism $f : A \to B$ in $\mathscr{A}$ we have

$$\alpha_B\big(H(f)(S(A))\big) = H'(f)(\alpha_A(S(A))) \subseteq H'(f)(S'(A)) \subseteq S'(B)\,,$$

and thus $H(f)(S(A)) \subseteq \alpha_B^{-1}(S'(B)) = S(B)$. We refer to the subfunctor S of H as the **preimage** of S' under α, and denote it by $\alpha^{-1}(S')$. We thus have

$$\alpha^{-1}(S')(A) = \alpha_A^{-1}(S'(A))$$

for every object A of $\mathscr{A}$. We get in this way a map

$$\alpha^{-1} : \mathscr{P}(S') \longrightarrow \mathscr{P}(S)\,.$$

This construction makes $\mathscr{P}$ functorial:

1. Let S be a subfunctor of H. We have the equalities
$$(1_H)^{-1}(S)(A) = 1_{H(A)}^{-1}(S(A)) = S(A)$$
for every object A of $\mathscr{A}$, and therefore $(1_H)^{-1}(S) = S$. Consequently, $(1_H)^{-1}$ is the identity map on $\mathscr{P}(H)$.
2. Let H'' be yet another functor from $\mathscr{A}$ to **Set**. Let S'' be a subfunctor of H'' and let
$$\alpha : H \Longrightarrow H'\,, \quad \alpha' : H' \Longrightarrow H''$$
be natural transformations. We have the chain of equalities
$$\begin{aligned} (\alpha' \circ \alpha)^{-1}(S'')(A) &= (\alpha' \circ \alpha)_A^{-1}(S''(A)) \\ &= (\alpha'_A \circ \alpha_A)^{-1}(S''(A)) \\ &= \alpha_A^{-1}\big((\alpha'_A)^{-1}(S''(A))\big) \\ &= \alpha_A^{-1}\big((\alpha')^{-1}(S'')(A)\big) \\ &= \alpha_A^{-1}((\alpha')^{-1}(S''))(A) \end{aligned}$$

for every object A of $\mathscr{A}$, and therefore the equality

$$(\alpha' \circ \alpha)^{-1}(S'') = \alpha^{-1}((\alpha')^{-1}(S'')) .$$

Consequently, $(\alpha' \circ \alpha)^{-1} = \alpha^{-1} \circ (\alpha')^{-1}$.

These properties tell us that we have constructed a contravariant functor

$$\mathscr{P} : [\mathscr{A}, \mathbf{Set}] \longrightarrow \mathbf{Set}$$

if the category $\mathscr{A}$ is small.

Naturality of the isomorphism $\mathrm{Sub}(H) \cong \mathscr{P}(H)$

We have previously constructed for every functor H from $\mathscr{A}$ to **Set** an isomorphism φ_H from $\mathscr{P}(H)$ to $\mathrm{Sub}(H)$. We will now show that the isomorphism φ_H is natural in H. If the category $\mathscr{A}$ is small, then this means that the two functors Sub and $\mathscr{P}$ from $\mathscr{A}$ to **Set** are isomorphic.

We need to check that for every two functors H and H' from $\mathscr{A}$ to **Set** and every natural transformation α from H to H' the following diagram commutes:

$$\begin{tikzcd}
\mathscr{P}(H') \arrow[r, "\varphi_{H'}"] \arrow[d, "\alpha^{-1}"'] & \mathrm{Sub}(H') \arrow[d, "\mathrm{Sub}(\alpha)"] \\
\mathscr{P}(H) \arrow[r, "\varphi_H"] & \mathrm{Sub}(H)
\end{tikzcd}$$

Let S' be an element of the top-left corner of this diagram, i.e., a subfunctor of H'. We give descriptions of the two resulting elements of $\mathrm{Sub}(H)$ and then show that these two elements are equal.

The subobject $\varphi_{H'}(S')$ of H' (an element of the tow-right corner of the diagram) is represented by (S', ι') where ι' is the natural transformation from S' to H' that is the inclusion map in each component. Let us denote the resulting element $\mathrm{Sub}(\alpha)([(S', \iota')])$ by $[(T, \theta)]$. This is a subobject of H, and it is uniquely determined by the fact that there exists a pullback diagram in $[\mathscr{A}, \mathbf{Set}]$ of the following form:

$$\begin{tikzcd}
T \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\theta"'] & S' \arrow[d, Rightarrow, "\iota'"] \\
H \arrow[r, Rightarrow, "\alpha"] & H'
\end{tikzcd}$$

The subfunctor $S \coloneqq \alpha^{-1}(S')$ of H (an element of the bottom-right corner of the diagram) is given by

$$S(A) = \alpha_A^{-1}(S'(A))$$

for every object A of $\mathbf{A}$. The subobject $\varphi_H(S)$ of H (an element of the bottom-right corner of the diagram) is represented by (S, ι) where ι denotes the natural transformation from S to X that is the inclusion map in each component.

To show the desired equality $[(S, \iota)] = [(T, \theta)]$ we need to show that there exists a pullback diagram of the form

$$\begin{tikzcd} S \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\iota"'] & S' \arrow[d, Rightarrow, "\iota'"] \\ H \arrow[r, Rightarrow, "\alpha"] & H' \end{tikzcd}$$

in $[\mathscr{A}, \mathbf{Set}]$. We know that limits in functor categories can be computed pointwise. It therefore suffices to show that there exists for every object A of $\mathscr{A}$ a pullback diagram of the form

$$\begin{tikzcd} S(A) \arrow[r] \arrow[d, "\iota_A"'] & S'(A) \arrow[d, "\iota'_A"] \\ H(A) \arrow[r, "\alpha_A"] & H'_A \end{tikzcd}$$

in $\mathbf{Set}$. This desired diagram simplifies as follows:

$$\begin{tikzcd} \alpha_A^{-1}(S'(A)) \arrow[r] \arrow[d, "\iota_A"'] & S'(A) \arrow[d, "\iota'_A"] \\ H(A) \arrow[r, "\alpha_A"] & H'_A \end{tikzcd}$$

We get this desired pullback diagram by choosing the upper horizontal arrow as the restriction of α_A, as seen in Proposition 6.G (page 224).

Special case: the category A has one object and is discrete

To better understand the problem at hand, let us first consider the special case that the category $\mathbf{A}$ is the one-object discrete category. The presheaf category $\hat{\mathbf{A}}$ is then just **Set**, and the functor $\mathscr{P}$ is just the usual contravariant power set functor.

Let $\mathbb{1} = \{1\}$ be a one-element set. For every set X, elements of X are the same as maps from $\mathbb{1}$ to X. For the subobject classifier Ω of **Set** we have therefore the isomorphisms

$$\Omega \cong \mathbf{Set}(\mathbb{1}, \Omega) \cong \mathscr{P}(\mathbb{1}) .$$

We turn this observation into motivation to define Ω as $\mathscr{P}(\mathbb{1})$.

We observe that $\Omega = \{\varnothing, \mathbb{1}\}$, which agrees with the usual definition of the subobject classifier of **Set**. But we won't need this explicit description of Ω, and will deliberately avoid it. Instead, our argumentation will rely on the following two properties of the set $\mathbb{1}$ and its element 1.

1. For every set X and every element x of X there exists a unique map e_x from $\mathbb{1}$ to X with $e_x(1) = x$.
2. The only subset of $\mathbb{1}$ containing 1 is $\mathbb{1}$.

In terms of intuition, the second property tells us that "the subset of $\mathbb{1}$ generated by the element 1 is $\mathbb{1}$".

In the following, let X be some set. We will describe how to abstractly construct the usual bijection between $\mathscr{P}(X)$ and $\mathbf{Set}(X, \Omega)$, and how to show that this bijection is natural in X.

Suppose first that S is a subset of X. For every element x of X we can pull back the subset S of X along the map e_x a subset of $\mathbb{1}$. We get in this way a map

$$\chi_S : X \longrightarrow \Omega , \quad x \longmapsto e_x^{-1}(S) .$$

(We observe that $\chi_S(x) = \mathbb{1}$ if $x \in S$ and $\chi_S(x) = \varnothing$ otherwise, so our abstract construction of the characteristic function χ_S agrees with the usual explicit definition.)

Suppose now that f is any map from X to Ω. The set $\mathbb{1}$ as a special subset, namely $\mathbb{1}$ itself. The set $U \coloneqq \{\mathbb{1}\}$ is then a subset of Ω, which we can pull back along f to the subset $f^{-1}(U)$ of X.

We will now check that the above two constructions between subsets of X and maps from X to Ω are mutually inverse, and also natural in X.

Let S be a subset of X. We have for every element x of X the chain of equivalences

$$\begin{aligned} x \in S &\iff e_x(1) \in S \\ &\iff 1 \in e_x^{-1}(S) = \chi_S(x) \\ &\iff \mathbb{1} = \chi_S(x) \\ &\iff x \in \chi_S^{-1}(U)\,, \end{aligned} \tag{6.23}$$

and therefore the equality $S = \chi_S^{-1}(U)$. For the step (6.23) we used that $\chi_S(x)$ is a subset of $\mathbb{1}$ containing 1, and that the only such subset is $\mathbb{1}$.

Let now f be a map from X to Ω and let $S := f^{-1}(U)$. To show the equality $f = \chi_S$ we need to show that $f(x) = \chi_S(x)$ for every element x of X. Both $f(x)$ and $\chi_S(x)$ are subsets of $\mathbb{1}$, so we need to show that for every element p of $\mathbb{1}$, we have $p \in f(x)$ if and only if $p \in \chi_S(x)$. The only element of $\mathbb{1}$ is 1, and we have the chain of equivalences

$$\begin{aligned} 1 \in f(x) &\iff f(x) = \mathbb{1} \\ &\iff x \in S \\ &\iff \chi_S(x) = \mathbb{1} \\ &\iff 1 \in \chi_S(x)\,. \end{aligned}$$

We have thus shown that indeed $f = \chi_S$.

To show the required naturality, we need to show that for every map

$$f : X \longrightarrow Y$$

the following diagram commutes:

$$\begin{tikzcd} \mathscr{P}(Y) \arrow[r, "\chi_{(-)}"] \arrow[d, "f^{-1}"'] & \mathbf{Set}(Y, \Omega) \arrow[d, "f^*"] \\ \mathscr{P}(X) \arrow[r, "\chi_{(-)}"] & \mathbf{Set}(X, \Omega) \end{tikzcd}$$

Let S be a subset of Y and let x be an element of X. Then $f \circ e_x = e_{f(x)}$ because

$$(f \circ e_x)(1) = f(e_x(1)) = f(x) = e_{f(x)}(1)\,,$$

and consequently

$$\begin{aligned} f^*(\chi_S)(x) &= (\chi_S \circ f)(x) \\ &= \chi_S(f(x)) \\ &= e_{f(x)}^{-1}(S) \\ &= (f \circ e_x)^{-1}(S) \\ &= e_x^{-1}(f^{-1}(S)) \\ &= \chi_{f^{-1}(S)}(x) \,. \end{aligned}$$

This shows that $f^*(\chi_S) = \chi_{f^{-1}(S)}$, so that the diagram commutes.

Special case: the category A has one object

Suppose now slightly more generally than before that the category $\mathbf{A}$ consists of only a single object $*$. For the monoid $M := \mathbf{A}(*, *)$, the presheaf category $\hat{\mathbf{A}}$ is then isomorphic to the category $\mathscr{M}$ of right M-sets. For every right M-set X the set $\mathscr{P}(X)$ consists of all M-subsets of X, and for every homomorphism of right M-sets $f : X \to Y$ the induced map $f^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X)$ is given by taking preimages in the usual sense.

For every right M-set X, the elements of X correspond to homomorphisms from M to X. For the subobject classifier Ω of $\mathscr{M}$ we have therefore the isomorphisms

$$\{\text{elements of } \Omega\} \cong \mathscr{M}(M, \Omega) \cong \mathscr{P}(M) \,.$$

We therefore define Ω as follows:

- The underlying set of Ω is $\mathscr{P}(M)$.
- For every element m of M, left multiplication with m is a homomorphism $\lambda_m : M \to M$, and therefore induces a map $\lambda_m^{-1} : \mathscr{P}(M) \to \mathscr{P}(M)$. This map is the action of m on Ω. More explicitly, we have

$$Sm = \lambda_m^{-1}(S) = \{n \in M \mid \lambda_m(n) \in S\} = \{n \in N \mid mn \in S\} \,. \tag{6.24}$$

We denote the set (6.24) as $(S : m)$, in accordance to the notation for quotient ideals in ring theory. We note that m is contained in S if and only if 1_M is contained in $(S : m)$.

More generally, if X is any right M-set, S is an M-subsets of X, and x is an element of X, then we will use the notation $(S : x)$ for the set $\{m \in M \mid xm \subseteq S\}$, so that

$$(S : x) \ni m \iff S \ni xm \,.$$

We note that x is contained in S if and only if 1_M is contained in $(S : x)$, if and only if $(S : x) = M$.

As in the previous special case, we will try not to work with the explicit elements of Ω. Instead, we will rely on the following two observations:

1. There exists for every right M-set X and every element x of X a unique homomorphism e_x from M to X with $e_x(1_M) = x$.
2. The only M-subsets of M that contains 1_M is M itself.

Let X be a right M-set and let S be an M-subsets of X. For every element x of X we can pull back S along the homomorphism e_x to an M-subsets of M. We have in this way a set-theoretic map

$$\chi_S : X \longrightarrow \Omega\,, \quad x \longmapsto e_x^{-1}(S)\,.$$

This map is more explicitly given by

$$\chi_S(x) = e_x^{-1}(S) = \{m \in M \mid e_x(m) \in S\} = \{m \in M \mid xm \in S\} = (S : x)$$

for every $x \in X$. As a by-product of the construction of χ_S, we see that $(S : x)$ is an M-subsets of M.

We claim that the map χ_S is already a homomorphism. To see this, we observe that $e_{xm} = e_x \circ \lambda_m$, since both e_{xm} and $e_x \circ \lambda_m$ are homomorphisms and

$$e_{xm}(1_M) = xm = e_x(1_M)m = e_x(m) = e_x(\lambda_m(1_M)) = (e_x \circ \lambda_m)(1_M)\,.$$

It follows that

$$\chi_S(xm) = e_{xm}^{-1}(S) = (e_x \circ \lambda_m)^{-1}(S) = \lambda_m^{-1}(e_x^{-1}(S)) = \chi_S(x)m\,.$$

Suppose conversely that f is any homomorphism from M to Ω. We note that M contains a very special M-subsets, namely M itself. Every element m of M is contained in M (surprise!), whence $Mm = (M : m) = M$. This tells us that M is a fixed point in Ω. Equivalently, $U := \{M\}$ is an M-subsets of Ω. By pulling back U along f, we arrive at the M-subsets $f^{-1}(U)$ of X.

We will now show that the above two constructions between M-subsets of X and homomorphism from X to Ω are mutually inverse and natural.

Let S be an M-subsets of X. We have for every element x of X the chain of equivalences

$$\begin{aligned} x \in S &\iff e_x(1_M) \in S \\ &\iff 1_M \in e_x^{-1}(S) = \chi_S(x) \\ &\iff \chi_S(x) = M \\ &\iff x \in \chi_S^{-1}(U) \end{aligned} \tag{6.25}$$

and therefore the equality $S = \chi_S^{-1}(U)$. We have used in the step (6.25) that $\chi_S(x)$ is an M-subsets of X containing 1_M, and that the only such subset is M itself.

Let f be a homomorphism from X to Ω. For the M-subsets $S := f^{-1}(U)$ of X we want to show that $f = \chi_S$. We need to show that $f(x) = \chi_S(x)$ for every element x of X. Both $f(x)$ and $\chi_S(x)$ are M-subsets of M, so we need to show that for every element m of M we have $m \in f(x)$ if and only if $m \in \chi_S(x)$. We have

$$(f(x) : m) = f(x)m = f(xm),$$

and therefore the chain of equivalences

$$\begin{aligned} m \in f(x) &\iff 1_M \in (f(x) : m) \\ &\iff 1_M \in f(xm) \\ &\iff f(xm) = M \\ &\iff xm \in S \\ &\iff m \in (S : x) \\ &\iff m \in \chi_S(x). \end{aligned}$$

This shows that indeed $f = \chi_S$.

It remains to show the naturality of the bijections $\chi_{(-)} : \mathscr{P}(X) \to \mathscr{M}(X, \Omega)$. For this, we need to show that for every homomorphism

$$f : X \longrightarrow Y$$

the following diagram commutes:

$$\begin{tikzcd} \mathscr{P}(Y) \arrow[r, "\chi_{(-)}"] \arrow[d, "f^{-1}"'] & \mathscr{M}(Y, \Omega) \arrow[d, "f^*"] \\ \mathscr{P}(X) \arrow[r, "\chi_{(-)}"] & \mathscr{M}(X, \Omega) \end{tikzcd}$$

Let S be an M-subsets of Y and let x be an element of X. Then $f \circ e_x = e_{f(x)}$ because

$$(f \circ e_x)(1_M) = f(e_x(1_M)) = f(x) = e_{f(x)}(1_M),$$

and consequently

$$\begin{aligned} f^*(\chi_S)(x) &= (\chi_S \circ f)(x) \\ &= \chi_S(f(x)) \\ &= e_{f(x)}^{-1}(S) \\ &= (f \circ e_x)^{-1}(S) \\ &= e_x^{-1}(f^{-1}(S)) \\ &= \chi_{f^{-1}(S)}(x). \end{aligned}$$

This shows that $f^*(\chi_S) = \chi_{f^{-1}(S)}$, which shows the required commutativity.

(a)

Suppose that Ω is a subobject classifier for $\hat{\mathbf{A}}$. For every object A of $\mathbf{A}$ we then have the isomorphisms

$$\Omega(A) \cong \hat{\mathbf{A}}(\mathrm{H}_A, \Omega) \cong \mathrm{Sub}(\mathrm{H}_A) \cong \mathscr{P}(\mathrm{H}_A).$$

(b)

We define the desired subobject classifier Ω of $\hat{\mathbf{A}}$ as the composite of the Yoneda embedding from $\mathbf{A}$ to $\hat{\mathbf{A}}$ (which is covariant) and the contravariant power set functor from $\hat{\mathbf{A}}$ to $\mathbf{Set}$. More explicitly, we have

$$\Omega(A) = \mathscr{P}(\mathrm{H}_A)$$

for every object A of $\mathbf{A}$, and for every morphism $f : B \to A$ in $\mathbf{A}$ the map

$$\Omega(f) := \mathrm{H}_f^{-1} : \mathscr{P}(\mathrm{H}_A) \longrightarrow \mathscr{P}(\mathrm{H}_B), \quad S \longmapsto \mathrm{H}_f^{-1}(S).$$

More explicitly, we have

$$\begin{aligned} \Omega(f)(S)(C) &= \mathrm{H}_f^{-1}(S)(C) \\ &= (\mathrm{H}_f)_C^{-1}(S(C)) \\ &= (f_*)^{-1}(S(C)) \\ &= \{g : C \to B \mid f_*(g) \in S(C)\} \\ &= \{g : C \to B \mid f \circ g \in S(C)\}. \end{aligned} \tag{6.26}$$

We will show in the following that the presheaf Ω is a subobject classifier for $\hat{\mathbf{A}}$, i.e., that $\hat{\mathbf{A}}(-, \Omega) \cong \mathrm{Sub}$. We have already seen that $\mathrm{Sub} \cong \mathscr{P}$, so we will show in the following that $\mathscr{P} \cong \hat{\mathbf{A}}(-, \Omega)$. We will use the following two properties of the presheaves H_A.

1. For every object A of $\mathbf{A}$ and every element x of $X(A)$ there exists a unique natural transformation ε_x from H_A to X with $\varepsilon_{x,A}(1_A) = x$.
2. For every object A of $\mathbf{A}$, the only subfunctor of H_A containing 1_A is H_A itself.

Let X be a presheaf on $\mathbf{A}$ and let S be a subfunctor of X. Let A be an object of $\mathbf{A}$ and let x be an element of $X(A)$. We can use the natural transformation ε_x to pull back the subfunctor S of X to the subfunctor $\varepsilon_x^{-1}(S)$ of H_A. We get in this way a set-theoretic map

$$\chi_{S,A} : X(A) \longrightarrow \mathscr{P}(\mathrm{H}_A) = \Omega(A) , \quad x \longmapsto \varepsilon_x^{-1}(S) .$$

More explicitly, we have

$$\begin{aligned} \chi_{S,A}(B) &= \varepsilon_x^{-1}(S)(B) \\ &= \varepsilon_{x,B}^{-1}(S(B)) \\ &= \{g \in \mathrm{H}_A(B) \mid \varepsilon_{x,B}(g) \in S(B)\} \\ &= \{g : B \to A \mid X(g)(x) \in S(B)\} . \end{aligned}$$

The resulting transformation χ_S from X to Ω is natural. To see this, we need to check that for every morphism

$$f : B \longrightarrow A$$

in $\mathbf{A}$ the following diagram commutes:

$$\begin{tikzcd} X(A) \arrow[r, "\chi_{S,A}"] \arrow[d, "X(f)"'] & \Omega(A) \arrow[d, "\Omega(f)"] \\ X(B) \arrow[r, "\chi_{S,B}"] & \Omega(B) \end{tikzcd}$$

We observe for every element x of $X(A)$ that

$$\varepsilon_{X(f)(x)} = \varepsilon_x \circ \mathrm{H}_f$$

since both sides are functors from H_B to X with

$$\begin{aligned}
\varepsilon_{X(f)(x),B}(1_B) &= X(f)(x) \\
&= X(f)(\varepsilon_{x,A}(1_A)) \\
&= \varepsilon_{x,B}\big((\mathrm{H}_A)(f)(1_A)\big) \\
&= \varepsilon_{x,B}(f) \\
&= \varepsilon_{x,B}\big((\mathrm{H}_f)_B(1_B)\big) \\
&= \big(\varepsilon_{x,B} \circ (\mathrm{H}_f)_B\big)(1_B) \\
&= (\varepsilon_{x,B} \circ \mathrm{H}_f)_B(1_B)\,.
\end{aligned}$$

It follows for every element x of $X(A)$ that

$$\begin{aligned}
\chi_{S,B}(X(f)(x)) &= \varepsilon_{X(f)(x)}^{-1}(S) \\
&= (\varepsilon_x \circ \mathrm{H}_f)^{-1}(S) \\
&= \mathrm{H}_f^{-1}(\varepsilon_x^{-1}(S)) \\
&= \Omega(f)(\chi_S(x))\,.
\end{aligned}$$

This shows the required commutativity.

Suppose now that α is a natural transformation from X to Ω. The functor H_A has itself as a subfunctor, and for every morphism $f : B \to A$ in $\mathbf{A}$ we have $\mathrm{H}_f^{-1}(\mathrm{H}_A) = \mathrm{H}_B$ because

$$\begin{aligned}
\mathrm{H}_f^{-1}(\mathrm{H}_A)(C) &= (\mathrm{H}_f)_C^{-1}(\mathrm{H}_A(C)) \\
&= \{g \in \mathrm{H}_B(C) \mid (\mathrm{H}_f)_B(g) \in \mathrm{H}_A(C)\} \\
&= \{g \in \mathbf{A}(C,B) \mid f \circ g \in \mathbf{A}(C,A)\} \\
&= \mathbf{A}(C,B) \\
&= \mathrm{H}_B(C)
\end{aligned}$$

for every object C of $\mathbf{A}$. This tells us that the presheaf Ω has a subfunctor U given by $U(A) = \{\mathrm{H}_A\}$ for every object A of $\mathbf{A}$. We can pull back the subfunctor U of Ω along the natural transformation α to the subfunctor $\alpha^{-1}(U)$ of X.

We will now check that these two constructions between subfunctors of X and natural transformations from X to Ω are mutually inverse, as well as natural.

Let S be a subfunctor of X. We observe for every object A of $\mathbf{A}$ and every element x of $X(A)$ the chain of equivalences

$$\begin{aligned}
& x \in S(A) \\
\iff\ & \varepsilon_{x,A}(1_A) \in S(A) \\
\iff\ & 1_A \in \varepsilon_{x,A}^{-1}(S(A)) = \varepsilon_x^{-1}(S)(A) = \chi_{S,A}(x)(A) \\
\iff\ & \chi_{S,A}(x) = \mathrm{H}_A \qquad (6.27) \\
\iff\ & x \in \chi_{S,A}^{-1}(\{\mathrm{H}_A\}) = \chi_{S,A}^{-1}(U(A)) = \chi_S^{-1}(U)(A) \,.
\end{aligned}$$

For the equivalence (6.27) we use that $\chi_{S,A}(x)$ is a subfunctor of H_A containing 1_A, and must therefore be all of H_A. This shows that $S(A) = \chi_S^{-1}(U)(A)$ for every object A of $\mathbf{A}$, and therefore $S = \chi_S^{-1}(U)$.

Let now α be a natural transformation from X to Ω and let $S = \alpha^{-1}(U)$. We want to check that $\alpha = \chi_S$. For this, we need to show that for every object A of $\mathbf{A}$ and every element x of $X(A)$ we have $\alpha_A(x) = \chi_{S,A}(x)$. Both $\alpha_A(x)$ and $\chi_{S,A}(x)$ are elements of $\Omega(A) = \mathscr{P}(\mathrm{H}_A)$, and therefore subfunctors of H_A. We hence need to show that $\alpha_A(x)(B) = \chi_{S,A}(x)(B)$ for every object B of $\mathbf{A}$. We have for every element f of $\mathrm{H}_A(B) = \mathbf{A}(B, A)$ the chain of equalities

$$\begin{aligned}
& f \in \alpha_A(x)(B) \\
\iff\ & 1_B \in \Omega(f)(\alpha_A(x))(B) \qquad (6.28) \\
\iff\ & \Omega(f)(\alpha_A(x)) = \mathrm{H}_B \qquad (6.29) \\
\iff\ & \alpha_B(X(f)(x)) = \mathrm{H}_B \qquad (6.30) \\
\iff\ & X(f)(x) \in \alpha_B^{-1}(\{\mathrm{H}_B\}) = \alpha_B^{-1}(U(B)) = \alpha^{-1}(U)(B) = S(B) \\
\iff\ & \varepsilon_{x,B}(f) \in S(B) \qquad (6.31) \\
\iff\ & f \in \varepsilon_{x,B}^{-1}(S(B)) = \varepsilon_x^{-1}(S)(B) = \chi_S(x)(B) \,,
\end{aligned}$$

and thus $\alpha_A(x)(B) = \chi_S(x)(B)$. For (6.28) we use the formula (6.26). The equality (6.27) holds because $\Omega(f)(\alpha_A(x))$ is a subfunctor of H_B that contains 1_B, and the only such subfunctor is H_B itself. For (6.30) we use the naturality of α, and for (6.31) we use that $X(f)(x)$ is precisely $\varepsilon_{x,B}(f)$.

To show the required naturality we need to check that for every natural transformation

$$\alpha : X \longrightarrow Y$$

between presheaves on $\mathbf{A}$ the following diagram commutes:

$$\begin{tikzcd}
\mathscr{P}(Y) \arrow[r, "\chi_{(-)}"] \arrow[d, "\alpha^{-1}"'] & \hat{\mathbf{A}}(Y, \Omega) \arrow[d, "\alpha^*"] \\
\mathscr{P}(X) \arrow[r, "\chi_{(-)}"] & \hat{\mathbf{A}}(X, \Omega)
\end{tikzcd}$$

Let S be a subfunctor of Y, let A be an object of $\mathbf{A}$ and let $x \in X(A)$, then

$$\begin{aligned}
\alpha^*(\chi_S)_A(x) &= (\chi_S \circ \alpha)_A(x) \\
&= (\chi_{S,A} \circ \alpha_A)(x) \\
&= \chi_{S,A}(\alpha_A(x)) \\
&= \varepsilon^{-1}_{\alpha_A(x)}(S) \\
&= (\alpha \circ \varepsilon_x)^{-1}(S) \\
&= \varepsilon_x^{-1}(\alpha^{-1}(S)) \\
&= \chi_{\alpha^{-1}(S),A}(x)\,,
\end{aligned}$$

therefore $\alpha^*(\chi_S)_A = \chi_{\alpha^{-1}(S),A}$, and thus $\alpha^*(\chi_S) = \chi_{\alpha^{-1}(S)}$. This shows the required commutativity.

(c)

We know from Corollary 6.2.11 that $\hat{\mathbf{A}}$ has all small limits. This entails that $\hat{\mathbf{A}}$ has finite limits. We have seen in Theorem 6.3.20 that $\hat{\mathbf{A}}$ is cartesian closed. We have seen in this exercise that $\hat{\mathbf{A}}$ is well-powered and has a subobject classifier.

This shows altogether that $\hat{\mathbf{A}}$ is a topos.

Appendix A

Proof of the general adjoint functor theorem

Exercise A.3

(a)

For every object B of $\mathscr{B}$, let i_B be the unique morphism from 0 to B.

Let for a moment $(C, (q_B)_B)$ be a cone on the identity functor $1_{\mathscr{B}}$. We have for every morphism $g : B \to B'$ in $\mathscr{B}$ the equality $g \circ q_B = q_{B'}$. By choosing the morphism g as i_B we find that

$$i_B \circ q_0 = q_B$$

for every object B of $\mathscr{B}$. This tells us that the entire cone $(C, (q_B)_B)$ is already uniquely determined by the single morphism $q_0 : C \to 0$.

Conversely, given any morphism $g : C \to 0$ in $\mathscr{B}$, we can pull back the cone $(0, (i_B)_B)$ along the morphism g to the cone $(C, (i_B \circ g)_B)$. We note that for $B = 0$ we have $i_B \circ g = i_0 \circ g = g$ since the morphism i_0 is necessarily the identity morphism of 0.

The above two constructions are mutually inverse. We thus find that a cone on the identity functor $1_{\mathscr{B}}$ with vertex C is the same as a morphism from C to 0.

Suppose now that $(C, (q_B)_B)$ and $(C', (q'_B)_B)$ are two cones on the identity functor $1_{\mathscr{B}}$. We find for every morphism f from C to C' that

$$\begin{aligned}
&f \text{ is a morphism of cones from } (C, (q_B)_B) \text{ to } (C', (q'_B)_B) \\
\iff\ & q'_B \circ f = q_B \text{ for every object } B \text{ of } \mathscr{B} \\
\iff\ & i_B \circ q'_0 \circ f = i_B \circ q_0 \text{ for every object } B \text{ of } \mathscr{B} \\
\iff\ & q'_0 \circ f = q_0 \,.
\end{aligned} \tag{A.1}$$

For the equivalence (A.1) we use again that i_0 is the identity morphism of 0.

We have thus found an isomorphism between the cone category $\mathbf{Cone}(1_{\mathscr{B}})$ and the slice category $\mathscr{B}/0$. This slice category has as a terminal object, namely $(0, 1_0)$. The corresponding cone $(0, (i_B \circ 1_0)_B) = (0, (i_B)_B)$ is therefore terminal in $\mathbf{Cone}(1_{\mathscr{B}})$. This means precisely that this cone is a limit cone.

(b)

We have for every morphism $g : B \to B'$ in $\mathscr{B}$ the equality

$$g \circ p_B = p_{B'} \,.$$

This entails that $p_B \circ p_L = p_B$ for every object B of $\mathscr{B}$, whence $p_B \circ p_L = p_B \circ 1_L$ for every object B of $\mathscr{B}$. Consequently, $p_L = 1_L$ by Exercise 5.1.36, part (a). It further follows for every object B of $\mathscr{B}$ that the morphism p_B is the only morphism from L to B: for every morphism g from L to B we have the equalities

$$p_B = g \circ p_L = g \circ 1_L = g \,.$$

Exercise A.4

(a)

For every two elements c and c' of C, there exists a morphism from c to c' if and only if $c \leq c'$. Therefore, S is weakly initial if and only if there exists for every element c of C an element s of S with $s \leq c$.

(b)

For every element c of C there exists an element s' of S with $s' \leq c$, whence

$$\bigwedge_{s \in S} s \leq s' \leq c \,.$$

Exercise A.5

(a)

For every object I of $\mathbf{I}$, an object $D(I)$ of $(A \Rightarrow G)$ is the same as a pair $(E(I), e_I)$ consisting of an object $E(I)$ of $\mathscr{B}$ and a morphism e_I from A to $G(E(I))$. For every morphism $u : I \to J$ in $\mathbf{I}$, a morphism $D(u)$ from $D(I)$ to $D(J)$ is then the same as a morphism $E(u)$ from $E(I)$ to $E(J)$ such that $G(E(u)) \circ e_I = e_J$.

The functoriality of the assignment D is precisely the functoriality of the assignment E. If this functoriality is satisfied, then $(A, (e_I)_) I$ is precisely a cone on $G \circ E$ with vertex A.

We hence find that a functor D from $\mathbf{I}$ to $(A \Rightarrow G)$ is the same as a functor E from $\mathbf{I}$ to $\mathscr{B}$ together with a cone $(A, (e_I)_I)$ on $G \circ E$. The two functors D and E are related through the projection functor P_A from $(A \Rightarrow G)$ to $\mathscr{B}$ via

$$E = P_A \circ D\,.$$

(b)

Let $\mathbf{I}$ be a small category and let D be a diagram in $(A \Rightarrow G)$ of shape $\mathbf{I}$. The resulting diagram $E := P_A \circ D$ in $\mathbf{B}$ admits a limit $(L', (p'_I)_I)$ since the category $\mathscr{B}$ is complete. We need to show the following two assertions:

1. There exists a unique cone $(L, (p_I)_I)$ on the original diagram D such that $L' = P_A(L)$ and $p'_I = P_A(p_I)$ for every object I of $\mathbf{I}$.
2. This cone $(L, (p_I)_I)$ is a limit cone.

We know from part (a) of this exercise that we may regard the diagram D in $(A \Rightarrow G)$ as the diagram E together with a cone of the form $(A, (e_I)_I)$ on $G \circ E$. To prove the first of the above two assertions, we make the following observations:

- Lifting the object L' of $\mathscr{B}$ to an object L of $(A \Rightarrow G)$ means choosing a morphism ℓ from A to $G(L')$ in $\mathscr{A}$, so that then $L = (L', \ell)$.
- Let I be some object of $\mathbf{I}$. There can be at most one lift of the morphism $p'_I : L' \to E(I)$ to a morphism $p_I : L \to D(I)$ because the projection functor P_A is faithful. More explicitly, the only possible lift is p'_I itself, but p'_I is a morphism from $L = (L', \ell)$ to $D(I) = (E(I), e_I)$ if and only if the diagram

$$\begin{tikzcd} & A \arrow[dl, "\ell"'] \arrow[dr, "e_I"] & \\ G(L') \arrow[rr, "G(p'_I)"] & & G(E(I)) \end{tikzcd}$$

commutes, i.e., if and only if $e_I = G(p'_I) \circ \ell$.

- Writing $p_I = p'_I$, the object L of $(A \Rightarrow G)$ together with the morphisms p_I is then automatically a cone for the diagram D. Indeed, we need for every morphism $u : I \to J$ the commutativity of the following diagram:

$$\begin{tikzcd} & L \arrow[dl, "p_I"'] \arrow[dr, "p_J"] & \\ D(I) \arrow[rr, "D(u)"] & & D(J) \end{tikzcd}$$

It suffices to check that this diagram commutes after applying the forgetful functor P_A, resulting in the following diagram:

$$\begin{tikzcd} & L' \arrow[dl, "p'_I"'] \arrow[dr, "p'_J"] & \\ E(I) \arrow[rr, "E(u)"] & & E(J) \end{tikzcd}$$

This diagram commutes because $(L', (p'_I)_I)$ is a cone on E.

As a consequence of these observations, to prove the first assertion, we need to show that there exists a unique morphism ℓ from A to $G(L')$ such that $G(p'_I) \circ \ell = e_I$ for every object I of $\mathbf{I}$.

We note that the object $G(L')$ of $\mathscr{A}$ together with the morphisms $G(p'_I)$ forms a cone on $G \circ E$ because $(L', (p'_I)_I)$ is a cone on E. We note that the cone $(G(L'), (G(p'_I))_I)$ in question is already a limit cone because the original cone $(L', (p'_I)_I)$ is a limit cone and the functor G is continuous. It follows that there exists a unique morphism ℓ from A to $G(L')$ with

$$G(p'_I) \circ \ell = e_I$$

for every object I of $\mathbf{I}$. The overall situation is depicted in Figure A.1.

We have thus proven the first assertion by explicitly construction the required cone $(L, (p_I)_I)$ on D.

Suppose now that $(C, (q_I)_I)$ is another cone on D. To prove the second assertion, we need to show that there exists a unique morphism f from C to L in $(A \Rightarrow G)$ such that $p_I \circ f = q_I$ for every object I of $\mathbf{I}$.

The object c is of the form $C = (C', c)$ for an object C' of $\mathscr{B}$ and a morphism $c : A \to G(C')$ in $\mathscr{A}$, and each morphism $q_I : C \to D(I)$ can we regarded as a morphism $q'_I : C' \to E(I)$ with

$$G(q'_I) \circ c = e_I \,. \tag{A.2}$$

$$
\begin{tikzcd}
A \arrow[rr, dashed, "\ell"] \arrow[dr, "e"'] & & G(L') \arrow[dl, "G(p')"] & & & L' \arrow[dl, "p'"] \\
& G \circ E & & \arrow[l, leftsquigarrow, phantom, ""] \rightsquigarrow \!\!\!\! & E & \\
& \text{in } \mathscr{A} & & & \text{in } \mathscr{B} &
\end{tikzcd}
$$

Figure A.1: Construction of ℓ.

We need to show that there exists a unique morphism f from C' to L' in $\mathscr{B}$ such that the following two properties hold:

1. $G(f) \circ c = \ell$, where both sides are morphisms from $G(C')$ to $G(L')$.
2. $p'_I \circ f = q'_I$ for every object I of $\mathbf{I}$.

Indeed, the first property is what it means for the morphism f from C' to L' to also be a morphism from $C = (C', c)$ to $L = (L', \ell)$, whereas the second property means precisely that $p_I \circ f = q_I$ for every object I of $\mathbf{I}$.

We know that $(C, (q_I)_I)$ is a cone on D, so by applying the projection functor P_A we find that $(C', (q'_I)_I)$ is a cone on E. It follows that there exists a unique morphism f from C' to L' satisfying the second property, as $(L', (p'_I)_I)$ is a limit cone on E.

To show the required equality $G(f) \circ c = \ell$, it suffices to show that

$$G(p'_I) \circ G(f) \circ c = G(p'_I) \circ \ell$$

for every object I of $\mathbf{I}$, thanks to Exercise 5.1.36, part (a) and because the cone $(G(C'), (G(p'_I))_I)$ is a limit cone. This required equality holds because

$$G(p'_I) \circ G(f) \circ c = G(p'_I \circ f) = G(q'_I) \circ c = e_I = G(p'_I) \circ \ell\,,$$

where we use the functoriality of G, the definition of f, identity (A.2), and the definition of ℓ.

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